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HIGHER

ARITHMETIC

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PREFACE.

1. The present work has been prepared with the belief that it will be of service to progressive teachers in American high schools, academies, and normal schools. As indicated by its title, it is intended for those who are taking up the subject a second time with the desire to review and extend the knowledge previously acquired. The purpose of the authors is more fully set forth in the following statement of some of the distinctive features of the book.

2. The applied problems refer to the ordinary commercial life of to-day, or they deal with elementary questions arising in the laboratory, or they are inserted for general information. The fact that tradition has furnished the schools with a mass of inherited puzzles which give a false notion of business, that in an age of science and invention these subjects have found no place in the arithmetics, and that the common graphic methods of representing statistics are not seen in the schools, has not deterred the authors from attempting to modernize the subject. At the same time they believe that the exercises will be found much more straightforward and simple than those with which the average text-book has so long been encumbered.

3. Problems in pure arithmetic in the high school are intended to furnish training in mathematical analysis. This is almost their only justification. Hence, the attempt has been made to lead the pupil to a clear understanding of

such subjects as the greatest common divisor, the multiplication and division of fractions, the square and cube roots, etc. To this end it has been necessary to resort to the literal notation. Most students will know enough algebra for this purpose, but for those who do not the necessary foundation can be laid in two or three lessons.

4. The work being intended as a review, it has not been thought necessary to attempt a definition of every arithmetical term which is employed. A table of the most common terms is given on p. vii, with definitions and etymologies.

5. The work being intended only for those teachers who recognize in arithmetic an instrument for mental training, no rules are given.

6. Teachers are urged to follow the suggestions laid down in the work, with respect to the omission of such chapters or portions of chapters as are not adapted to their pupils, and to change the sequence to suit their own tastes. It is hardly necessary to suggest that only a portion of the exercises should be attempted by any one class, the advantage of changing from term to term being evident.

7. While the authors have not failed to consult the leading French, German, Italian, and English arithmetics, they have taken but few problems from these sources. To Day's "Electric Light Arithmetic" (London, 1887) they are, however, indebted for several exercises.

8. In the work in mensuration several figures have been taken from the authors' "Plane and Solid Geometry" (Boston, Ginn & Co.). For the scientific treatment of that subject and for additional exercises, teachers are referred to the text-book mentioned.

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TABLE OF CONTENTS.

DEFINITIONS AND ETYMOLOGIES.

The following list includes such definitions as are apt to be needed for reference, together with pronunciations and etymologies. The latter are those given by the Century Dictionary.

Key. L. Latin, G. Greek, F. French, ML. Mediaeval Latin, LL. Low Latin, AS. Anglo-Saxon, ME. Middle English, dim. diminutive, fem. feminine.

a *fat*,	ā *fate*,	ä *far*,	â *fall*,	ȧ *ask*,	ã *fare*,
e *met*,	ē *mete*,	ė *her*,	i *pin*,	ī *pine*,	o *not*,
ō *note*,	ö *move*,	ô *nor*,	u *tub*,	ū *mute*,	u̇ *pull*.

s̤ as in *leisure*.

A single dot under a vowel indicates its abbreviation.

A double dot under a vowel indicates that the vowel approaches the short sound of *u*, as in *put*.

The numerals refer to pages in the arithmetic.

Abscissa (ab-sis′ä̤). L. cut off. A certain line used in determining the position of a point in a plane. 68.

Abstract (ab′strakt). L. *abstractus*, drawn away. An abstract number is a number not designated as referring to any particular class of objects. *E.g.*, 7, as distinguished from 7 ft. which is a *concrete number*.

Addition (a-dish′o̤n). L. *addere*, to increase. The uniting of two or more numbers in one sum.

Ampere (am-pãr′). Term adopted by the Electric Congress at Paris, 1881. Name of French electrician, André Marie Ampère, d. 1836. The unit employed in measuring the strength of an electric current. 101.

Analysis (a̤-nal′i-sis). G. a loosing, a resolution of a whole into its parts. A form of reasoning from a whole to its parts.

Antecedent (an-tẹ-sē′-de̤nt). L. *antecedere*, to go before. The first of two terms of a ratio.

Antilogarithm (an-ti-log′a̤-rithm). G. *anti-*, opposite to, + logarithm. The number corresponding to a logarithm. 117.

Approximation (a̤-prok-si-mā′sho̤n). L. *ad*, to, + *proximare*, to come near. (a) A continual approach to a true result. (b) A result so near the truth as to be sufficient for a given purpose.

Are (ãr or är). L. *area*, a piece of level ground. A square dekameter, or 119.6 sq. yds.

Area (ā′rē̤-ạ̈). See Are. The superficial contents of any figure or surface.

Arithmetic (a̤-rith′me̤-tik; as adj., ar-ith-met′ik). G. *arithmos*, number. The theory of numbers, the art of computation, and the applications of numbers to science and business.

Arithmetic series. See Series. 101.

Associative law (a̤-sō′shi-ā̤-tiv). L. *ad*, to, + *sociare*, to join. The law which states that certain operations give the same result whether they first unite two quantities A and B, and then unite the result to a third quantity C; or first unite B and C, and then unite the result to A, the order of the quantities being preserved.

Average (av′e̤-rāj). Etymology obscure. The result of adding several quantities and dividing the sum by the number of quantities.

Avoirdupois (av″o̤r-dū-poiz′). F. *aver*, goods, + *de*, of, + *pois*, weight. A system of weight in which 1 lb. = 16 oz. = approximately 7000 troy grains.

Axiom (ak′si-o̤m). G. *axioma*, a requisite, a self-evident principle. A simple statement, of a general nature, so obvious that its truth may be taken for granted.

Bank discount (bangk). ML. *bancus*, bench. See Discount.

Base (bās). LL. *bassus*, low. (a) The line or surface forming that part of a figure on which it is supposed to stand. (b) The base of a system of logarithms is the number which, raised to the power indicated by the logarithm, gives the number to which the logarithm belongs. (c) In percentage, the number which is multiplied by the rate to produce the percentage.

Bond (bond). AS. *bindan*, to bind. An obligation, under seal, to pay money. It may be issued by a government, a railway corporation, a private individual, etc.

Broker (brō′kėr). Originally, one who manages. An agent.

Brokerage (brō′kėr-ā̤j). The fee or commission given to a broker.

Cancel (kan′sel). L. *cancelli*, a lattice. Originally, to draw lines across a calculation. To strike out or eliminate as a common factor in the terms of a fraction, a common term in the two members of an equation, etc.

Cast out nines. 14.

Characteristic (kar″ak-te̤-ris′tik). G. *characterizein*, to designate. 112.

Check. ME. *cheker*, a chess board. To verify; that is, to mark off as having been examined.

Circle (sir′kl). L. *circulus*, dim. of *circus* (G. *kirkos*), a ring. A plane figure whose periphery is everywhere equally distant from a point within it, the center.

Circulating decimal (sėr′kū̦-lāt-ing). 109.

Circumference (sėr-kum′fe̤-re̤ns). L. *circum*, around (see Circle), + *ferre*, to bear. The line which bounds a circle.

Cologarithm (kō-log′a̤-rithm). 118.

Commission (ko̤-mish′o̤n). L. *com-*, together, + *mittere*, to send. The act of intrusting; hence, a fee paid to one who is intrusted.

Common denominator (kom′o̤n). L. *communis*, general, universal. A denominator common to two or more fractions.

Common fraction. See Fraction. A fraction in which both terms are written out in full, as distinguished from a decimal fraction.

Common multiple. See Multiple. A multiple of two or more expressions is a common multiple of those expressions.

Commutative law (ko̤-mū′ta̤-tiv). L. *com-*, intensive, + *mutare*, to change. The law which states that the order in which elements are combined is indifferent.

Complement (kom′plē̤-me̤nt). L. *com-*, intensive, + *plere*, to fill. A number added to a second number to complete a third.

Complex fraction (kom′pleks). L. *com-*, together, + *plectere*, to weave. (See Fraction.) A common fraction whose numerator or denominator contains a common fraction.

Composite number (ko̤m-poz′it). L. *com-*, together, + *ponere*, to put. A number which can be exactly divided by a number exceeding unity.

Compound interest (kom′pound). See Composite. 145.

Compound number. 54.

Concrete number (kon′krēt or ko̤n-krēt′). L. *concretus*, grown together. A number which specifies the unit, as 3 ft. In the case of 3 ft., 3 is strictly the number and 1 ft. is the unit; it is, however, convenient to designate 3 as an abstract number and 3 ft. as a concrete number.

Cone (kōn). G. *kōnos*, a cone. The elementary form considered in arithmetic is a solid generated by the revolution of a right-angled triangle about one of its sides as an axis.

Consequent (kon′sē̤-kwent). L. *com-*, together, + *sequi*, to follow. The latter of the two terms of a ratio.

Consignee (kon-sī-nē′). L. *com-*, together, + *signare*, to seal. One who has the care or disposal of goods upon consignment.

Consignor (ko̤n-sī′no̤r or kon-si-nôr′). A person who makes a consignment.

Corporation (kôr-pō̤-rā′sho̤n). L. *corporare*, to form into a body, from *corpus*, a body. An artificial person created by law from a group of natural persons and having a continuous existence irrespective of that of its members.

Coupon bond (kö′pon). F. *couper*, to cut. A bond, usually of a state or corporation, for the payment of money at a future day, with severable tickets or coupons annexed, each representing an instalment of interest, which may be conveniently cut off for collection as they fall due.

Cube (kūb). G. *kubos*, a die, a cube. (a) A regular solid with six square faces. (b) To raise to the third power. (c) The third power of a number.

Cube root. The cube root of a perfect third power is one of the three equal factors of that power. A number which has not a perfect third power has not three equal factors. It is, however, said to have a cube root to any required degree of approximation. Thus, the cube root of n to 0.1 is that number of tenths whose cube differs from n by less than the cube of any other number of tenths.

Cylinder (sil′in-der). G. *kylindros*, from *kyliein*, to roll. The elementary form considered in arithmetic is a solid generated by the revolution of a rectangle about one of its sides as an axis.

Date line (dāt′ līn). The fixed boundary line between neighboring regions where the calendar day is different.

Decimal (des′i-ma̤l). L. *decem*, ten. Pertaining to ten.

Denominator (dē̤-nom′i-nā-to̤r). L. *denominare*, to name. 27.

Diagonal (dī-ag′ō̤-na̤l). G. *dia*, through, + *gonia*, corner, angle. A line through the angles of a figure, but not lying in its sides or faces.

Difference (dif′e-re̤ns). L. *differens*, different. The difference between two numbers is the number which added to either will produce the other. In arithmetic it is usually taken as the number which added to the smaller will produce the larger.

Digit (dij′it). L. *digitus*, finger. The number represented by any one of the ten symbols 0, 1, 2, ····· 9. The term is more often used to designate one of the ten symbols mentioned.

Directly proportional. 97.

Discount (dis′kount). L. *dis*, away, + *computare*, to count. An allowance or deduction made from the customary or normal price, or from a sum due or to be due at a future time. Bank discount is simple interest paid in advance, reckoned on the amount of a note.

Distributive law (dis-trib′ū-tiv). 14.

Dividend (div′i-dend). L. *dividere*, to divide. (a) A number or quantity to be divided by another called the divisor. (b) A division of profits to be distributed proportionately among stockholders.

Divisibility (di-viz-i-bil′i-ty). The quality of being divisible without a remainder. Usually applied in speaking of abstract integers.

Division (di-vizh′on). 29.

Divisor (di-vī′zor). See Dividend.

Draft (dråft). AS. *dragan*, to draw. A writing directing the payment of money on account of the drawer. 155, 156.

Drawee of a draft. One on whom the order is drawn.

Drawer of a draft. One who draws the order for the payment of money.

Duty (dū′ti). Due + ty. Sum of money levied by a government upon goods imported from abroad.

Equal (ē′kwal). L. *aequalis*, equal. Having the same value.

Equation (ē-kwā′shon). A proposition asserting the equality of two quantities and expressed by the sign = between them. In algebra, an equality which exists only for particular values of certain letters called the unknown quantities.

Equilateral (ē-kwi-lat′e-ral). L. *aequus*, equal, + *latus*, side. Having all the sides equal.

Evolution (ev′ō-lū′shon). L. *evolvere*, to unroll. The extraction of roots from powers.

Exchange (eks-chānj′). ML. *ex*, out, + *cambiare*, to change. The transmission of the equivalent of money from one place to another, such equivalent being redeemable in the money of the place to which it is sent.

Exponent (eks-pō′nent). L. *exponere*, to set forth, indicate. A symbol placed above and at the right of another symbol (the base) to denote that the latter is to be raised to a power. For general meaning, see pages 32, 33.

Extremes (eks-trēmz′). L. *extremus*, outermost. The first and last terms of a proportion or of any other related series of terms.

Face (fās). L. *facies*, face. The principal sum due on a note, bond, policy, etc.

Factor (fak′tọ̈r). L. *facere*, to do. One of two or more numbers which when multiplied together produce a given number.

Fraction (frak′shọ̈n). L. *frangere*, to break. 26.

Fractional unit. One of the equal parts of unity.

Geometric series (jē-ọ̄-met′rik). G. *geometria*, geometry. 104.

Grace (grās). L. *gratus*, dear. 141.

Gram (gram). 59.

Greatest common divisor of two or more integers is the greatest integer which will divide each of them without a remainder.

Hypotenuse (hī-pot′e-nūs). G. *hypo*, under, + *teinein*, to stretch. The side of a right-angled triangle opposite the right angle.

Improper fraction. A common fraction whose terms are positive, and whose numerator is not less than its denominator.

Index notation. 1.

Insurance (in-shör′ạ̈ns). OF. *enseurer*, to insure. A contract by which one party for an agreed consideration undertakes to compensate another for loss on a specific thing.

Integer (in′tẹ̄-jėr). L., a whole number.

Interest (in′tėr-est). L. *interest*, it concerns. A sum paid for the use of money.

Inversely proportional (in-vėrs′li). 97.

Involution (in-vọ̄-lū′shọ̈n). L. *involvere*, to roll up. Multiplication of a quantity into itself any number of times.

Isosceles (ī-sos′e-lēz). G. *isos*, equal, + *skelos*, leg. Having two sides equal.

Least common denominator. The least denominator which can be common to two or more common fractions.

Least common multiple. The least integer which is a multiple of two or more given integers.

Lever (lev′ėr or lē′vėr). L. *levare*, to raise. A bar acted upon at different points by two forces which severally tend to rotate it in opposite directions about a fixed point called the fulcrum.

Liter (lē′tėr). G. *litra*, a pound. The unit of capacity in the metric system; a cubic decimeter. 59.

Logarithm (log′ạ̈-rithm). G. *logos*, word, + *arithmos*, number. The exponent of the power to which a number called the base (in the common system, 10) must be raised to produce a number.

Longitude (lon′ji-tūd). L. *longus*, long. The angle at the pole between two meridians, one of which, the Prime Meridian, passes through some conventional point and from which the angle is measured.

Mantissa (man-tis′ạ̈). L., something left over. 112.

Maturity (mạ̄-tū′rị̈-ti), L. *maturus*, mature. The time when a note or bond becomes due.

Means (mēnz). The second and third terms of a proportion.

Measure (mezh′ụ̈r). L. *mensura*, measure. (a) A unit or standard adopted to determine the length, volume, or other quantity of some other object. (b) The determination of quantity by the use of a unit.

Mensuration (men-ṣū-rā′shọn). The science of measuring.

Meridian (mē-rid′i-ạn). L. *meridianus*, belonging to mid-day. A semi-circumference passing through the poles.

Meter (mē′ter). G. *metron*, measure. Unit of length in the metric system. 59.

Metric (met′rik). 59.

Mikron (mī′kron). G. *mikros*, small. Millionth part of a meter.

Minuend (min′ụ̄-end). L. *minuere*, to lessen. The number from which another number is subtracted.

Mixed number. The sum of an integer and a fraction.

Multiple (mul′ti-pl). L. *multus*, many, + *plus*, akin to E. fold. A number produced by multiplying an integer by an integer.

Multiplicand (mul′ti-pli-kand). A number multiplied by another number called the multiplier.

Multiplication (mul-ti-pli-kā′shọn). 29.

Multiplier. See Multiplicand.

Net proceeds (net prō′sēds). The sum left from the sale of a note or other piece of property after every charge has been paid.

Notation (nọ̄-tā′shọn). L. *notare*, to mark. A system of written signs of things and relations used in place of common language.

Note (nōt). L. *nota*, mark. A written or printed paper acknowledging a debt and promising payment.

Number (num′bėr). L. *numerus*, a number. An abstract number is the ratio of one quantity to another of the same kind. See Abstract, Concrete.

Numeration (nū-mẹ-rā′shọn). L. *numerare*, to count. The art of reading numbers.

Numerator (nū′mẹ-rā-tọr). The number, in a common fraction, which shows how many parts of a unit are taken.

Ohm (ōm). Named after G. S. Ohm, a German electrician. The unit of electrical resistance. 101.

Ordinate (ôr′di-nạt). L. *ordinare*, to order. 68.

Parallelepiped (par-ạ-lel-e-pip′ed or pī′ped). G. *parallelos*, parallel, + *epipedon*, plane. A prism whose bases are parallelograms.

Parallelogram (par-a-lel′ọ-gram). G. *parallelos*, parallel, + *gramma*, line. A quadrilateral whose opposite sides are parallel.

Par value. L. *par*, equal. Face value.

Per capita (per cap′i-tạ̈). L., by the head.

Per cent. L., by the hundred. 126.

Percentage (per-sent′ạ̄j). (a) That portion of arithmetic which involves the taking of per cents. (b) The result from multiplying a number (called the base) by a certain rate per cent.

Policy (pol′i-si). ML. *politicum*, a register. A contract of insurance.

Poll tax (pōl). ME. *poll*, head. A tax sometimes levied at so much per head of the adult male population.

Premium (prē′mi-um). L. *praemium*, profit. (a) Amount paid to insurers as consideration for insurance. (b) Amount above par at which stocks, drafts, etc., are selling.

Present worth of a sum. An amount which placed at interest at a given rate will amount to that sum in a given time.

Prime number (prīme). L. *primus*, first. An integer not divisible without a remainder by any integer except itself and unity. Two integers are prime to one another when they have no common divisor except unity.

Prime meridian. See Longitude.

Principal (prin′si-pạl). L. *princeps*, first. A capital sum lent on interest.

Prism (prizm). G. *priein*, to saw. A solid whose bases are parallel congruent polygons and whose sides are parallelograms.

Problem (prob′lem). G. *problema*, a question proposed for solution.

Product (prod′ukt). L. *pro-*, forward, + *ducere*, to lead. The result from multiplying one number by another.

Proper fraction. A common fraction whose terms are positive, and whose numerator is less than its denominator.

Proportion (prō-pọ̄r′shọn). L. *pro*, before, + *portio*, share. 96.

Pyramid (pir′ạ-mid). G. *pyramis*, a pyramid. A solid contained by a plane polygon as base, and other planes meeting in a point.

Pythagorean theorem (pi-thag′ō̤-rē-a̤n). A theorem first proved by Pythagoras. 68.

Quadrilateral (kwod-rī̤-lat′ē̤-ral). L. *quatuor*, four, + *latus*, a side. A four-sided plane figure.

Quantity (kwon′ti̤-ti). L. *quantus*, how much? The being so much in measure or extent.

Quotient (kwō′she̤nt). L. *quotiens*, how many times? The number which taken with the divisor as a factor produces the dividend.

Radius (rā′di-us). L., rod. A line from the center of a circle to the circumference.

Rate per cent. 127.

Ratio (rā′shio). L., a reckoning. 87.

Reciprocal numbers (rē-sip′rō̤-ka̤l). L. *reciprocus*, alternating. Two numbers which multiplied together make unity.

Rectangle (rek′tang-gl). L. *rectus*, right, + *angulus*, angle. A quadrilateral all of whose angles are right angles.

Rectangular parallelepiped. A parallelepiped all of whose faces are rectangles.

Reduction (rē̤-duk′sho̤n). L. *re-*, back, + *ducere*, to bring. Changing the denomination of numbers. Reduction ascending, changing to a higher denomination, as from 144 inches to 12 feet. Reduction descending, changing to a lower denomination.

Registered bond (rej′is-tėrd). A bond bearing the name of the owner, the name and residence being registered on the books of the corporation issuing the bond.

Remainder (rē̤-mān′der). L. *re-*, back, + *manere*, to stay. The same as Difference.

Root (röt or rüt). ME. *roote*, root. The root of a number is such a number as, when multiplied into itself a certain number of times, will produce that number. For more extended definition, see Cube Root, Square Root.

Series (sē′rēz or sē′ri-ēz). L. *series*, a row, from *serere*, to join together. 104.

Share (shãr). AS. *sceran*, to cut. One of the whole number of equal parts into which the capital stock of a corporation is divided.

Significant figure (sig-nif′i-ka̤nt). L. *signum*, a sign, + *facere*, to make. The succession of figures in the ordinary notation of a number, neglecting all the ciphers between the decimal point and the figure not a cipher nearest to the decimal point.

Solid (sol′id). L. *solidus*, firm. Any limited portion of space.

Specific gravity (spẹ-sif′ik grav′i-ti). L. *species*, kind, + *facere*, to make. 92.

Sphere (sfēr). G. *sphaera*, a ball. A solid bounded by a surface whose every point is equidistant from a point within the solid, called the center of the sphere.

Square (skwãr). L. *quatuor*, four. (a) An equilateral rectangle. (b) The second power of a number. (c) To raise a number to the second power.

Square root. 34.

Standard time (stan′dạ̈rd). ML. *standardum*, standard. A system of uniform time for a given section of country. 85.

Stere (stãr). G. *stereos*, solid. A cubic meter; 35.31 cu. ft.

Stock (stok). AS. *stoc*, post, trunk. The share capital of a corporation.

Subtraction (sub-trak′shọ̈n). L. *sub*, under, + *trahere*, draw. The operation of finding the difference between two numbers. See Difference.

Subtrahend (sub′trạ̄-hend). The number subtracted from the minuend.

Sum (sum). L. *summa*, the highest part. See Addition.

Surd (sėrd). L. *surdus*, deaf. A number not expressible as the ratio of two integers.

Surface (sėr′fạ̄s). L. *superficies*, the upper face. The bounding or limiting parts of a solid.

Terms of a fraction (tėrms). L. *terminus*, limit. The numerator and denominator together.

Theorem (thē′ọ̄-rem). G. *theorema*, a sight. A statement of a truth to be demonstrated.

Thermometer (thėr-mom′e-tėr). G. *therme*, heat, + *metron*, measure. An instrument by which temperature is measured.

Trapezoid (trạ̄-pē′zoid). G. *trapeza*, table, + *eidos*, form. A quadrilateral having two parallel sides.

Triangle (trī′ang-gl). L. *tres*, three, + *angulus*, angle. A three-sided plane figure.

Unit (ū′nit). L. *unus*, one. Any standard quantity by the representation and subdivision of which any other quantity of the same kind is measured.

Volt (vōlt). From Volta, an Italian physicist. The unit of electromotive force. 101.

Volume (vol′ūm). L. *volvere*, to roll round. Solid contents.

SYMBOLS AND ABBREVIATIONS.

THE following are used in this work, and are inserted here for reference. Other symbols are explained as needed. For historical note, see p. 43.

E.g., Latin *exempli gratia*, for example.

I.e., Latin *id est*, that is.

ex., exercise.

ax., axiom.

th., theorem.

g.c.d., greatest common divisor.

l.c.m., least common multiple.

π, Greek letter π, symbol for 3.14159 $\cdots\cdots$

$\because$, since.

$\therefore$, therefore.

%, per cent, hundredth, hundredths. See p. 126.

$+$, plus, symbol of addition and of positive numbers.

$-$, minus, symbol of subtraction and of negative numbers.

$\pm$, plus or (and) minus.

$\times$, $\cdot$, and absence of sign between letters, denote multiplication.

$\div$, /, and fractional form denote division.

$:$, ratio, a special form of division.

$=$, equal or equals.

$\doteq$, approaches as a limit. See p. 108.

$::$, sometimes used in proportion as a sign of equality.

$>$, is (or are) greater than.

$<$, is (or are) less than.

$\neq$, does (or do) not equal.

$\ngtr$, is (or are) not greater than.

$\nless$, is (or are) not less than.

$\cdots\cdots$, and so on.

a^{-2}, a^{-1}, a^0, a^1, a^2, $\cdots\cdots$ a^n, indicate powers. See pp. 2, 32.

$\sqrt{}$ and the exponent $\frac{1}{2}$ indicate square root. See p. 33.

$\sqrt[n]{}$ and the exponent $\frac{1}{n}$ indicate nth root.

( ), $\overline{}$, symbols of aggregation.

log, colog, antilog, see pp. 111–118.

For abbreviations of common measures, see pp. 52, 53. For abbreviations of metric measures, see pp. 60, 61.

n' is read n-prime.

f_1 " f sub one, or f-one.

t_n " t sub n, or (if there can be no misunderstanding) t-n.

HIGHER ARITHMETIC.

CHAPTER I.

Notation and the Fundamental Operations.

I. Writing and Reading Numbers.

The universal notation among civilized nations at the present time is the Hindu or Arabic, the symbols of which, except the zero, originated in India before the beginning of the Christian era, and seem to have been the initial letters of the early numerals. The system derives its intrinsic importance, however, from the zero, which renders possible the distinctive feature known as *place value.* Thus, in the number 302, the 3 stands for hundreds because it is in hundreds' place, a fact which could not be conveniently indicated without the symbol 0 or its equivalent. The zero appeared about the fifth century A.D., and somewhat later the Arabs, coming from the East after the conquest of Spain, brought the new system with them. About the year 1200 these Hindu numerals began to be known in Christian Europe, but it was not until the fifteenth century that they were generally taught and used. The decimal point appeared about the opening of the seventeenth century, and through its influence the subject of arithmetic, both pure and applied, has materially changed. The extent of this change may be seen in the number of cases in which the decimal fraction is used to-day.

The Roman numerals were in common use in Europe prior to the introduction of the Hindu system. As now written they have changed considerably from the form used at the beginning of the Christian era. In America they are at present employed chiefly in numbering the chapters of books, and hence are rarely used beyond one or two hundred. The old custom of printing the number of the year in Roman notation on the titlepages of books has practically ceased.

In modern science numbers are often used which contain several zeros, for the reason that absolute accuracy of measurement is generally impossible. Thus, it is said that the distance from the earth to a certain star is 21,000,000,000,000 miles, but the distance even to within a billion miles is quite unknown. Similarly, the length of a wave of sodium light is said to be 0.0005896 of a millimeter, but the seventh decimal place is doubtful and the subsequent ones are unknown. The naming of these numbers is a matter of little importance, and the writing of them in full is usually unnecessary. Scientists often resort to an *index notation*, in which an integer, sometimes followed by a decimal, is multiplied by a power of 10. Thus, 21,000,000,000,000 may be written 2.1×10^{13}, or 21×10^{12}. And since 10^{-1} means 0.1, and 10^{-2} means 0.01, etc., therefore 0.00000274 may be written 2.74×10^{-6}.

Since the index notation is now so extensively used in science, and since the limit of necessary counting in financial affairs is met in the billion or trillion, no elaborate system of naming numbers is practically used. Attention should be paid, however, to the proper reading of the numbers in common use, types of which are given in the examples on p. 3. Thus, 123.4567 should be read, one hundred twenty-three and four thousand five hundred sixty-seven ten-thousandths.

Exercises. 1. What are the various names given to the symbol 0?

2. Read the numbers 0.0002, 0.00004, 0.400.

3. Also, 0.123, 100.023.

4. Also, 0.1246, 1200.0046.

5. In the numbers XV and 15, why are the Hindu characters said to have a place value and the Roman not?

6. Express in the index notation the numbers in the following statements:

(*a*) In a cubic centimeter of air there is 0.00001114 of a grain of water.

(*b*) A centimeter is 0.0000062138 of a mile.

(*c*) The distance to the sun is 93,000,000 miles.

(*d*) The distance from the sun to Neptune is 2,788,800,000 miles.

7. Express in the common notation the numbers in the following statements:

(*a*) The distance from the equator to the pole is 39.377786×10^7 inches.

(*b*) The earth's polar radius is 6.35411×10^8 centimeters.

8. A syndicate is to bid on some government bonds; to how many decimal places should they express their bid per $100 if they bid for $10,000 worth? $1,000,000 worth? $25,000,000 worth? $100,000,000 worth?

II. Checks.

A check on an operation is another operation whose result tends to verify the result of the first. *E.g.*, if $11 - 7 = 4$, then $7 + 4$ should equal 11; this second result, 11, verifies the first result, 4.

The verification is usually incomplete. If, as is said, "the result does not check," there must be an error in (1) the original operation, (2) the check, or (3) both. If, on the other hand, the result does check, there may have been an error in one operation which just balanced the other. Hence a check makes it more or less improbable that an error remains undiscovered. The secret of accurate computation largely lies in the knowledge and the continued use of proper checks.

III. Addition.

In adding the annexed column, the computer should say to himself, "Five, fourteen, *eighteen;* one, seven, *fourteen;* nine, *twelve,*" thus omitting all superfluous words. Bookkeepers, whose business leads to rapid addition, omit much that would seem necessary to the student, and not infrequently add two columns at once, a power gained only by practice in their profession.

$$\begin{array}{r} 374 \\ 69 \\ 805 \\ \hline 1248 \end{array}$$

The most practical way to check addition is to perform the operation a second time. Experience shows, however, that if the operation is performed in the same way the mind tends to fall into the same error. Hence it is better to add the numbers in the reverse order.

It is unnecessary to give many exercises in addition, since the student who finds himself in need of practice can easily prepare them.

Exercises. 1. As an exercise, it is convenient to prepare a table of multiples of some number in this manner: Write any number, as 4197, on paper, and the same number on a small card, $\boxed{4197}$; place the card above the number and add, thus giving 2×4197; slide the card down and add again, thus giving 3×4197, and so on to 10×4197, when the work checks if the result is 41,970.

2. In the series of numbers 0, 1, 1, 2, 3, 5, 8, 13, 21, ⋯⋯, each number after the first two is obtained by adding the two preceding numbers; calculate the 30th number of the series.

3. Add 15, 214, 3962, 9984, 9785, 6037. Check.

4. Add 201, 76, 435, 7726, 8687, 8812, 8453, 1162. Check.

5. What is the sum of the first two odd numbers, *i.e.*, $1 + 3$? of the first 3? of the first 4? of the first 10? of the first 20? From these results, what would be *inferred* as to the sum of the first 100 odd numbers?

6. Explain why $234 + 859$ is 1000 more than the difference between 234 and 141. Also why $4396 + 8501$ is 10,000 more than the difference between 4396 and 1499.

IV. Subtraction.

There are three common methods of subtraction. In the annexed example, we may say,

(1) 5 from 14, **9**; 2 from 2, **0**; 3 from 12, **9**;	1234
(2) 5 from 14, **9**; 3 from 3, **0**; 3 from 12, **9**;	325
(3) 5 and **9**, 14; 2 and 1 and **0**, 3; 3 and **9**, 12.	909

Each of these three methods is easily understood. The first is the simplest of explanation, and hence it is generally taught to children. The second is slightly more rapid than the first. But the third, familiar to all as the common method of "making change," is so much more rapid than either of the others that it is recommended to all computers.

Since subtraction is the inverse of addition, the simplest check is the addition of the subtrahend and difference; the sum should equal the minuend.

Exercises. 1. If not entirely familiar with the third method above given, use it with enough problems to become so, and state the reason of its advantage in rapidity over the others.

2. In subtracting 34,256 from 100,000, show that the subtraction can easily be made from the left by taking each digit from 9, except the 6, which must be taken from 10.

3. As in Ex. 2, show how to subtract 27,830 from 100,000; also 948,900 from 1,000,000.

In Exs. 2 and 3, the results are called the *complements* of the given numbers, because they complete the next higher power of 10.

4. Show that the difference between 1234 and 5612 may be found by adding the complement of 1234 to 5612 and then subtracting 10,000.

5. Does $a - b$ equal $a + (10 - b) - 10$? Does $a - b$ equal $a + (10^n - b) - 10^n$? Show that this proves that the method of subtraction given in Ex. 4 is general.

6. Is there any advantage in subtracting by means of adding the complement of the subtrahend in a case like $521 - 173$?

7. Solve the problem $6872 - 4396 + 342 - 896 - 243 + 750$ by adding the proper complements and finally subtracting the proper powers of 10. Is there any advantage in using the method of complements in this case?

V. Multiplication.

The multiplication table is usually learned to 10×10. This is all that is necessary for practical purposes. It is often convenient, however, to perform quickly multiplications with certain larger numbers. There are many rules for such operations, most of them of little practical value. Some are, however, quite useful, and these are given.

To multiply by 5, 25, $33\frac{1}{3}$, 125 is the same as to multiply by $\frac{10}{2}$, $\frac{100}{4}$, $\frac{100}{3}$, $\frac{1000}{8}$. It is easier to multiply by $\frac{10}{2}$, that is, to add a cipher and divide by 2, than to multiply by 5, and similarly for the other cases. Also, to divide by 5 is the same as to divide by $\frac{10}{2}$ or to multiply by $\frac{2}{10}$, and it is easier to multiply by 2 and divide by 10 than to divide by 5. These short processes have especial value because 5%, 12.5%, 25%, $33\frac{1}{3}$%, 50% are common rates in business and scientific problems.

To multiply by 9 is to multiply by $(10 - 1)$. Hence, it suffices to annex a cipher and subtract the original number from the result, a method not always more convenient than ordinary multiplication.

To multiply by 11 is to multiply by $(10 + 1)$. Hence, it suffices to annex a cipher and add the original number; or, what is more convenient, to add the digits as in the following example: 11×248,

$$8 + 0 = \mathbf{8},\ 4 + 8 = \mathbf{12},\ 2 + 4 + 1 = \mathbf{7},\ 2 + 0 = \mathbf{2};$$

therefore, the result is 2728.

To multiply by numbers differing but little from 10^n. For example, to multiply by 997 is to multiply by $(10^3 - 3)$; that is, to annex three ciphers and subtract 3 times the multiplicand.

$$\begin{aligned} \textit{E.g.},\ 995 \times 1474 &= 1{,}474{,}000 - \tfrac{1}{2} \text{ of } 14{,}740 \\ &= 1{,}474{,}000 - 7370 = 1{,}466{,}630. \end{aligned}$$

When two factors lie between 10 and 20 the product is readily found as follows:

$$14 \times 16 = 10\,(14 + 6) + 4 \times 6 = 224.$$

To prove that this process is general:

(1) Any number from 10 to 19 may be represented by

$$10 + a,\ 10 + b,\ \cdots\cdots$$

(2) $(10 + a)(10 + b) = 100 + \underline{10\,a} + 10\,b + ab,$

$$= 10\,(10 + a + b) + ab.$$

To square numbers ending in 5. While not as practical as the problems already given, this has some value. The method is illustrated by 65^2; it is merely necessary to say,

$$6 \times 7 = 42,\ 5 \times 5 = 25,\ \therefore 65^2 = 4225.$$

To prove that this process is general:

(1) Any number ending in 5 may be represented by $10\,a + 5$, where a may equal 0, 1, 2, ····· 9, 10, ·····

(2) $$(10\,a + 5)^2 = 100\,a^2 + 100\,a + 25,$$
$$= 100\,a\,(a + 1) + 25.$$

(3) That is, the result ends in 25, and the number of hundreds is $a\,(a + 1)$.

Applications of the formulas for $(a + b)^2$, $(a + b)(a - b)$. In a few problems there is an advantage in recalling the identities

$$(a + b)^2 = a^2 + 2\,ab + b^2,$$
$$(a + b)(a - b) = a^2 - b^2.$$

For example, $62^2 = 3600 + 240 + 4 = 3844.$

$$23 \times 17 = (20 + 3)(20 - 3) = 391.$$

Oral Exercises. 1. Multiply 12,564 by 5; by 25.

2. Multiply 4239 by $0.33\frac{1}{3}$.
3. Multiply 5148 by 5; by 2.5; by $33\frac{1}{3}$; by 12.5.
4. Square 25, 35, 45, ····· 95, 105, 115.
5. Multiply 14 by 19; 17 by 15; 16 by 18; 12 by 19; 13 by 17.
6. Multiply 22 by 99; by 98; by 97.
7. Multiply 12,345 by 11.
8. Explain the short process of dividing by $33\frac{1}{3}$; by 125.

Arrangement of work. In multiplication, there is no practical advantage in beginning with the lowest order of units of the multiplier; in fact, there is a decided advantage in beginning with the highest order, as is clearly apparent in approximations in multiplication. The arrangement of work would then be as shown in the annexed example. Since $20 \times 0.009 = 0.18$, the position of the decimal point is at once known, and the rest of the process is apparent.

437.189
26.93
8743.78
2623.134
393.4701
13.11567
11773.49977

Approximations in multiplication are frequently desired. This arises from the fact that perfect measurements are rarely possible in science, and that results beyond two or three decimal places are seldom desired in business. Thus, if the radius of a wheel is known only to 0.001 inch, it is not possible to compute the circumference with any greater degree of accuracy; hence, labor would be wasted in seeking a product to more than three decimal places. In such cases all unnecessary work should be omitted, as in the annexed example. This represents the multiplication in the solution of the following problem: To find the circumference of a steel shaft of which the diameter is found by measurement to be 10.48 centimeters. Since it was practical to carry the original measurement only to 0.01 centimeter, the result need be sought only to 0.01. In order to be sure that the result is correct to 0.01, the partial products are carried to 0.001, and in multiplying by 0.04, for example, the effect of the fourth decimal place on the third is kept in mind.

10.48
3.1416
31.44
1.048
0.419 .
0.010 . .
0.006 . . .
32.92

Checks. The best check on multiplication is the "casting out of nines," explained in Chap. II.

Exercises. 1. Multiply 42.3531 by 3.1416, carrying the result to 0.001, that is, "correct to 0.001."

2. Multiply 126 by 0.3183, correct to 0.1.

3. If $\sqrt{2} = 1.414 \cdots\cdots$, find the values of $4.324\sqrt{2}$, $0.057\sqrt{2}$, and $8.346\sqrt{2}$, correct to 0.01.

4. If $\sqrt{3} = 1.732 \cdots\cdots$, find the values of $2.35\sqrt{3}$, $42.89\sqrt{3}$, and $0.869\sqrt{3}$, correct to 0.1.

5. Find $\sqrt{2}\cdot\sqrt{3}$, correct to 0.01.

6. Find the interest on $1525.75 for one year at $6\frac{1}{2}\%$, correct to $0.01.

7. If the circumference of a circle is 3.1416 times the diameter, and if the radius of the earth is found by measurement to be 6,378,249.2 meters, find the circumference to the necessary number of decimal places, the earth being supposed spherical. Express the result in the index notation.

8. Similarly, find the circumference of a wheel the diameter of which is found by measurement to be 6.3 ft.

9. Similarly, find the circumference of a shaft the radius of which is found by measurement to be 4.32 in.

10. If 1 yard is found by measurement to be 0.914 of a meter, find, to the necessary number of decimal places, the number of meters in 23.463 yds.

11. Similarly, if 1 meter is found by measurement to be 3.28 feet, find the number of feet in 3.476 meters.

12. If the number of cubic units of volume in a sphere is $\frac{4}{3}$ of 3.1416 times the third power of the number of units of radius, find the volume of a sphere whose radius is found by measurement to be 3.27.

13. Find the value of $9 \times 12{,}345{,}678 + 9$; of $9 \times 1{,}234{,}567 + 8$; of $9 \times 123{,}456 + 7$.

14. Find the value of $9 \times 98{,}765{,}432 + 0$; of $9 \times 9{,}876{,}543 + 1$; of $9 \times 987{,}654 + 2$.

15. Find the value of $8 \times 123{,}456{,}789 + 9$; of $8 \times 12{,}345{,}678 + 8$; of $8 \times 1{,}234{,}567 + 7$.

16. Show that $8{,}212{,}890{,}625^2$ terminates in 8,212,890,625.

17. If a railway uses a freight car belonging to another company it pays the owner 0.6 of a cent a mile for the distance run; during one year the freight cars of the United States were used in this way over a total distance of about 12,000,000,000 miles. How much rental did they earn their owners?

18. Prove that to multiply by 625 one may move the decimal point four places to the right and then divide by 2 four times.

VI. Division.

In division there is an advantage in placing the quotient above the dividend. The decimal point is then easily fixed, although it is unnecessary to carry it through the work.

```
         10.478
31416)329200.
      314160
       15040.0
       12566.4
        2473.60
        2199.12
         274.480
         251.328
          23.152  remainder.
```

In case a decimal point appears in both dividend and divisor, it is better to multiply each by such a power of 10 as shall make the divisor integral. Thus, in the case of $32.92 \div 3.1416$ it is better to multiply both numbers by 10^4, and divide 329200 by 31416, as above.

The work may be further abridged by omitting partial products and decimal points, keeping only the partial dividends, as here shown.

```
         10.478
31416)329200
       15040
        24736
         27448
          23152
```

It is advisable in extended cases of division to prepare a table of multiples of the divisor, as in the following division of 4,769,835 by 291:

1	291
2	582
3	873
4	1164
5	1455
6	1746
7	2037
8	2328
9	2619

```
  16391
4769835
1859
 1138
  2653
   345
    54  remainder.
```

Approximations in division. In division as in multiplication, approximations are often necessary. For example, if the circumference of a shaft is found by measurement to be 32.92 centimeters and it is required to know the diameter, it would be a waste of time to attempt to find the diameter beyond 0.01. Since 10's divided by 10,000's $<$ 0.01's, the last two figures of the dividend will not affect the quotient within two decimal places and hence may be neglected. Hence, also, the divisor may be considered as 3142 and may be continually contracted. The process is apparent by first examining the complete form in the example below. The student should note how much better for practical purposes the last form is than the others, and he is recommended to become so familiar with it as to use it in all cases where only approximate results are required.

$$\begin{array}{rrlll}
 & 10.48 & & & \\
31416) & 329200 & & & \\
 & \underline{3142} & = \text{approximately} & & 10 \times 3141(6) \\
 & 150 & & & \\
 & \underline{126} & = & \text{"} & 0.4 \text{ of } 314(16) \\
 & 24 & & & \\
 & \underline{24} & = & \text{"} & 0.08 \text{ of } 31(416)
\end{array}$$

$$\begin{array}{rr}
 & 10.48 \\
31416) & 329200 \\
 & 150 \\
 & 24
\end{array}$$

Checks. The best check on division is the "casting out-of nines" explained in Chap. II. Since the dividend equals the product of the quotient and divisor, plus the remainder if any, the work may also be checked by one multiplication and one addition.

Exercises. 1. Divide 42,856,731,275,834 by 574,238, correct to the tens' place.

2. Divide 100 by 3.1416, correct to 0.01.

3. Divide 5,080,240 by 40,467, correct to 0.1.

4. Divide 1 by 3.14159, correct to 0.001.

5. The population of British India is about 225,000,000, and the area is about 965,000 square miles; what is the average population per square mile, to the nearest unit?

6. In a certain year it cost $357,231,799 to pay the expenses of the United States government, which was $5.346 to each person; what was the population in that year, to the nearest 1000?

7. In a certain year the revenue of the United States government was $403,080,983, which was $6.577 to each person; what was the population in that year, to the nearest 1000?

8. A hectare is 2.471 acres, and a liter is 61.027 cubic inches; express the rainfall of 1 liter per hectare in cubic inches per acre, correct to 0.1.

9. Knowing that the circumference of a circle is $2 \times 3.14159 \times$ the radius, find, correct to 0.1, how many revolutions a mill-wheel 12 feet in diameter makes per minute when the speed of the periphery is 6 feet per second.

10. How many revolutions per mile are made by a locomotive drive-wheel 4.5 feet in diameter? (Correct to units.)

11. 84.25 liters of water are drawn through a pipe every 4.5 minutes from a tank containing 23,711 liters; how long will it take to empty the tank? (Correct to 1 minute.)

12. If 41 liters of water weigh as much as 51 liters of alcohol, and 1 liter of water weighs 1 kilogram, how much will 1 liter of alcohol weigh? (Correct to 0.01 kilogram.)

13. The horse-power of an engine is usually calculated by the formula $\frac{p \cdot l \cdot a \cdot n}{33{,}000}$, where p, l, a, n are abstract numbers representing the *p*ressure in pounds per square inch on the piston, the *l*ength of the stroke in feet, the *a*rea of the piston in square inches, and the *n*umber of strokes per minute. Calculate the horse-power, to the nearest unit, of each of these engines:

(*a*) $p = 20$, $l = 6$, $a = 400$, $n = 60$.

(*b*) $p = 8$, $l = 11$, $a = 3600$, $n = 40$.

(*c*) $p = 25$, $l = 3$, $a = 100$, $n = 90$.

(*d*) $p = 18$, $l = 5$, $a = 200$, $n = 50$.

Axioms. There are a number of general statements of mathematics the truth of which may be taken for granted. Such statements are called *axioms.*

The following are the axioms most frequently used in elementary arithmetic and algebra, the first, second, third, sixth, and seventh being especially important :

1. *Numbers which are equal to the same number or to equal numbers are equal to each other.*

If $5 - x = 3$, and $1 + x = 3$, then $5 - x = 1 + x$.

2. *If equals are added to equals, the sums are equal.*

If $x - 2 = 7$, then $x - 2 + 2 = 7 + 2$, or $x = 9$.

3. *If equals are subtracted from equals, the remainders are equal.*

If $x + 2 = 9$, then $x = 7$.

4. *If equals are added to unequals, the sums are unequal in the same sense.*

If $x + 2 > 8$, then $x + 2 + 5 > 8 + 5$.

5. *If equals are subtracted from unequals, the remainders are unequal in the same sense.*

If $x + 5 < 16$, then $x < 11$.

6. *If equals are multiplied by equal numbers, the products are equal.*

If $\frac{x}{3} = 6$, then $x = 18$.

7. *If equals are divided by equals, the quotients are equal.*

If $2x = 6$, then $x = 6 \div 2 = 3$.

8. *Like powers of equal numbers are equal.*

If $x = 5$, then $x^2 = 25$.

9. *Like roots of equal numbers are arithmetically equal.*

If $x^2 = 25$, then $x = 5$. (From algebra it should be remembered that $x = \pm 5$.)

Fundamental laws of elementary operations. There are also certain fundamental laws of number, the strict proof of which for cases involving fractions, surds, negative numbers, etc., is properly a part of algebraic analysis. Since their treatment is too advanced for an elementary work, their validity is here assumed. They are not, however, axioms, because they are not generally taken for granted in mathematics.

These laws, as well as the axioms on p. 13, are so frequently used in subsequent discussions that their formal statement is necessary. They are as follows:

1. *The associative law for addition and subtraction,* that the terms of an expression may be grouped in any way desired.

E.g., $a + b - c + d = a + (b - c) + d = (a + b) - (c - d) = \cdots\cdots$

2. *The commutative law for addition and subtraction,* that these operations may be performed in any order desired.

E.g., $a + b - c + d = a - c + d + b = d - c + a + b = \cdots\cdots$

3. *The associative law for multiplication and division,* that these operations may be grouped in any way desired.

E.g., $a \cdot b \cdot c \div d \cdot e = a \cdot (b \cdot c) \div (d \div e) = a \cdot b \cdot (c \div d) \cdot e = \cdots\cdots$

4. *The commutative law for multiplication and division,* that these operations may be performed in any order desired.

E.g., $a \cdot b \div c = a \div c \cdot b = b \div c \cdot a = \cdots\cdots$

Of course it is absurd to say that *72 times \$3* is the same as *\$3 times 72*, since the latter has no meaning by the common idea of multiplication. All that the commutative law asserts is that $72 \cdot \$3 = 72 \cdot 3 \cdot \$1 = 3 \cdot 72 \cdot \$1 = 3 \cdot \$72$.

5. *The distributive law for multiplication and division,* that $m(a - b) = ma - mb$, and that $\frac{a-b}{m} = \frac{a}{m} - \frac{b}{m}$.

CHAPTER II.

Factors and Multiples.

I. Tests of Divisibility.

In speaking of factors and multiples, only integers are considered. Thus, 2 and 3.5 are not considered factors of 7, nor is 9 considered a multiple of 2, although $4\frac{1}{2} \times 2 = 9$.

In practical computations, cancellation enters extensively, requiring a knowledge of the factors of numbers. There is no general process for determining large factors, and hence elaborate factor tables have been prepared for those who have extensive computations to make. But there are simple methods for determining small factors, methods not only valuable in a practical way, but also for the logic involved in their consideration.

Fundamental theorems. The theory of factors and multiples depends largely on two theorems :

I. *A factor of a number is a factor of any of its multiples.*

1. Let n be any number of which f is a factor, q being the quotient of $\frac{n}{f}$.

2. Then, since $\frac{n}{f} = q$,

3. $\therefore \frac{n}{f} + \frac{n}{f} + \cdots\cdots m$ times $= q + q + \cdots\cdots m$ times, Ax. 2

4. Or $\frac{mn}{f} = mq$. That is, f is a factor of mn, a multiple of n.

II. *A factor of each of two numbers is a factor of the sum or the difference of any two of their multiples.*

1. If $\frac{n}{f} = q$, and $\frac{n'}{f} = q'$,
2. Then $\frac{an}{f} = aq$, and $\frac{bn'}{f} = bq'$. Th. 1
3. Suppose $an > bn'$,
4. Then $\frac{an \pm bn'}{f} = aq \pm bq'$. Axs. 2, 3

That is, f is a factor of $an + bn'$ and of $an - bn'$, the sum and the difference of two multiples of n and n'.

Tests of divisibility.

I. *2 is a factor of a number if it is a factor of the number represented by its last digit, and not otherwise.*

1. Any number has the form $10a + b$, where b has any value from 0 to 9 inclusive, and a has any value from and including 0.
(*E.g.*, $7036 = 10 \times 703 + 6$, and $3 = 10 \times 0 + 3$.)
2. 2 is a factor of 10, and hence of $10a$. Th. 1
3. $\therefore$ 2 is a factor of $10a + b$ if it is a factor of b. Th. 2
4. Otherwise 2 is not a factor of $10a + b$, for $\frac{10a + b}{2} = 5a + \frac{b}{2}$, and $\frac{b}{2}$ is not an integer.

II. *4 is a factor of a number if it is a factor of the number represented by its last two digits, and not otherwise.*

Any number has the form $100a + 10b + c$, where, etc. The proof, which is similar to that of I, is left for the student.

III. *8 is a factor of a number if it is a factor of the number represented by its last three digits, and not otherwise.*

The proof, which is similar to the proofs of I and II, is left for the student.

IV. *5 is a factor of a number if it is a factor of the number represented by its last digit, and not otherwise.*

The proof, which is similar to that of I, is left for the student.

V. *9 is a factor of a number if it is a factor of the sum of the numbers represented by its digits, and not otherwise.*

1. Any number may be represented by

$$a + 10\,b + 100\,c + 1000\,d + 10{,}000\,e + \cdots\cdots$$

where a represents the units' digit, b the tens', $\cdots\cdots$

(*E.g.*, in 7024, $a = 4$, $b = 2$, $c = 0$, $d = 7$, $e = 0$, $\cdots\cdots$)

2. Or by $a + 9\,b + b + 99\,c + c + 999\,d + d + 9999\,e + e + \cdots\cdots$

3. Or by $9\,b + 99\,c + 999\,d + \cdots\cdots + a + b + c + d + \cdots\cdots$

4. Or by a multiple of 9, plus the sum of the numbers represented by the digits.

5. $\therefore$ 9 is a factor if it is a factor of the latter, and not otherwise.

VI. *3 is a factor of a number if it is a factor of the sum of the numbers represented by its digits, and not otherwise.*

The proof, the first three steps of which are the same as the first three steps of V, is left for the student.

VII. *6 is a factor of a number if 2 is a factor of the number represented by the last digit, and if 3 is a factor of the sum of the numbers represented by its digits, and not otherwise.*

For, since $6 = 2 \times 3$, if the number is divisible by 2 and 3, it is divisible by 6.

VIII. *11 is a factor of a number if it is a factor of the difference between the sums of the numbers represented by the odd and the even orders of digits, and not otherwise.*

E.g., 14,619 is divisible by 11, since $(1 + 6 + 9) - (4 + 1)$ is divisible by 11.

Proof. 1. Any number may be represented by

$$a + 10\,b + 100\,c + 1000\,d + 10{,}000\,e + \cdots\cdots$$

2. Or by $a + 11\,b - b + 99\,c + c + 1001\,d - d + 9999\,e + e + \cdots\cdots$

3. Or by $11\,b + 99\,c + 1001\,d + 9999\,e + \cdots\cdots$

$$+ (a + c + e + \cdots\cdots) - (b + d + \cdots\cdots)$$

4. Or by a multiple of 11, plus the difference between the sums of the numbers represented by the odd and the even orders of digits.

5. $\therefore$ 11 is a factor if it is a factor of the latter, and not otherwise.

IX. *There is no simple method of testing divisibility by 7.*

Exercises. 1. State a short test for divisibility by 12 and prove that it is correct.

2. Similarly for 15.

3. Similarly for 16.

4. Similarly for 18.

5. Prove that 4 is a factor of a number if it is a factor of the sum of the units' digit and twice the tens', or of the difference between them.

6. Prove that 8 is a factor of a number if it is a factor of the sum of the units' digit, and twice the tens', and four times the hundreds'.

7. Is 823 a prime number? Why is it unnecessary to try factors above 23 in answering this question? What are the prime factors of 13,168?

8. Prove that every even number is of the form $2n$, every odd number of the form $2n \pm 1$, and every number not divisible by 3 of the form $3n \pm 1$.

9. Prove that every prime number above 3 is of the form $6n \pm 1$.

10. Prove that one of any two consecutive even numbers is divisible by 4. Hence, show that 24 is a factor of the product of any three consecutive numbers if the middle one is odd.

11. Prove that the product of any three consecutive numbers is divisible by 6, and that the product of any five consecutive numbers is divisible by 120.

12. By squaring $10a + b$, and then letting b have the various values 0, 1, 2, ····· 9, prove that every square number is of the form $5n$ or $5n \pm 1$. Hence, show that an integer cannot be a square number unless, when divided by 5, there is a remainder of 0, 1, or 4.

13. What numbers below 19 (excluding 7, 13, 17) are factors of 472,396,890,163?

14. Similarly for 729,876,312 (excluding 7, 13, 14, 17).

15. Prove that because 7 and 13 are factors of 1001 they are factors of 2002, 3003, ····· 99,099, 100,100, 101,101 ·····

16. Reduce to lowest terms, by cancellation, the fractions $\frac{693}{17017}$, $\frac{3465}{9702}$, and $\frac{1105}{6875}$.

17. Reduce to lowest terms, by cancellation, the fractions $\frac{3003}{104104}$ and $\frac{3003003}{104104104}$.

18. Reduce to lowest terms, by cancellation, the fractions $\frac{1683}{41345377}$, $\frac{590}{1078}$, and $\frac{123453}{422763}$.

19. Reduce to lowest terms, by cancellation, the fractions $\frac{6237}{14641}$ and $\frac{5103}{131769}$.

II. Casting out Nines.

The practical method of determining whether or not a number is divisible by 9 is as follows: Add the numbers represented by the digits and reject ("cast out") each 9 as it is reached; the resulting number represents the remainder.

Thus, to determine whether 124,763 is divisible by 9, say "3, **9** (reject it), 7, 11 (reject 9), 2, 4, 5"; therefore, the remainder is 5. Since the order of adding is immaterial, the eye usually groups the 9's at once and the remainder is easily detected.

Check on addition by casting out nines. Since numbers are always multiples of 9 plus some remainder, they are of the type $9m+r$. By adding numbers of this type, the sum is a multiple of 9, plus the sum of the excesses. Therefore, *the excess of nines in a sum is equal to the excess in the sum of the excesses.*

$$\begin{array}{l} 9m \;\;+r \\ 9m'+r' \\ 9m''+r'' \\ \;\;\vdots \qquad \vdots \\ \hline 9(m+m'+\cdots\cdots) \\ \quad +r+r'+\cdots\cdots \end{array}$$

This check is too long to be of any value in addition alone; it is, however, a necessary part of the check on division.

Check on multiplication by casting out nines. Any two numbers may be represented by $9m+r$, $9m'+r'$. Their product is then represented by $9^2mm'+9(mr'+m'r)+rr'$, that is, by a multiple of 9, plus rr'. (Prove this by multiplying $9m+r$ by $9m'+r'$.) Since the excess of nines in this product is the excess in rr', therefore *the excess of nines in any product equals the excess in the product of the excesses.*

E.g.,
$$\begin{array}{r} 38 = \;\;\;4\times 9+2 \\ 51 = \;\;\;5\times 9+6 \\ \hline 1938 = 215\times 9+3 \end{array}$$

3 is the excess in 2 × 6, that is, the excess in the product of the two excesses, 2 and 6.

Check on division by casting out nines. Since the dividend equals the product of the quotient and divisor, plus the remainder, *the excess of nines in the dividend equals the excess in the sum of the excess in the product of the excesses of divisor and quotient, plus the excess in the remainder.*

E.g., 561,310,123 $\div$ 7654 = 73,335, with a remainder of 4033.

$\therefore$ 561,310,123 = 7654 $\times$ 73,335 + 4033.

$\therefore$ the excess in 561,310,123, which is 4, equals the excess in 4 $\times$ 3, which is 3, plus the excess in 4033, which is 1.

Of course these checks fail to discover any error not affecting the excess of nines, as an interchange of digits, the addition of 9, etc., but such errors are rare.

Exercises. 1. What is the remainder after dividing 4,236,987 by 9? also 147,362? also 140,076,923?

2. If 9,342,813 and 6,123,345 were added, would the sum be divisible by 9?

3. Is 4,238,964,108 divisible by 9? by 3? by 6? by 11?

4. If 12,345,601 were multiplied by 47,623,092 would the product be divisible by 3? by 9? by 11?

5. Determine mentally which of the following products are incorrect by the test of casting out nines: (1) 41,376 $\times$ 87,147 = 3,605,794,272; (2) 11.1 $\times$ 307.05 = 3418.255; (3) 30,303 $\times$ 300,065 = 9,192,869,695; (4) 121^3 = 1,771,561; (5) 444 $\times$ 31 = 13,764.

6. Show that multiplication may also be checked by casting out threes. Apply this method to the products in Ex. 5.

7. Similarly by casting out elevens.

8. Is the product of the numbers 4398, 14,765, 900,427, and 2002 divisible by 2? by 3? by 4? by 5? by 6? by 8? by 9? by 11?

9. Determine mentally which of the following quotients are incorrect by the test of casting out nines: (1) 865,432 $\div$ 2171 = 398 with 1374 remainder; (2) 866,555 $\div$ 5843 = 148 with 1891 remainder; (3) 4000 $\div$ 23 = 173 with 21 remainder; (4) 9012 $\div$ 173 = 52 with 16 remainder.

10. Show that division may also be checked by casting out threes. Apply this method to the quotients in Ex. 9.

III. Greatest Common Divisor.

The subject of greatest common divisor has lost much of its practical value since the decimal fraction came into quite general use, during the eighteenth century. Formerly it was necessary to reduce a fraction like $\frac{1829}{2419}$ to its lowest terms before it could be conveniently used in operations, *e.g.*, added to another fraction. For this purpose, the greatest common divisor (here 59) was found and cancelled from each term. And since the greatest common divisor of such numbers is not easily found by inspection, the long division process (called from Euclid, who used it about 300 B.C., the "Euclidean method") was employed. This Euclidean method is now rarely met in practice, but as a piece of logical reasoning it is so valuable as to deserve a place in the review of arithmetic.

The method of factoring may be illustrated by the following example: Required the greatest common divisor of 9801, 33,759, and 121,968.

1. $9801 = 3^4 \times 11^2$.
2. $33,759 = 3^2 \times 11^2 \times 31$.
3. $121,968 = 3^2 \times 11^2 \times 2^4 \times 7$.
4. And since the greatest common divisor is the greatest factor common to the three numbers, it is $3^2 \times 11^2$, or 1089.

Exercises. 1. Find, by factoring, the g.c.d. of 153, 891, and 1008.

2. Also of 32,760, 1170, and 1573.
3. Also of 720, 336, and 1736.
4. Also of 837, 1134, and 1377.
5. Also of 187, 253, and 341.
6. Also of 1331 and 4,723,598.
7. Also of 231, 165, 451, 4004, and 2827.
8. Also of 117, 143, 221, 338, and 650.
9. Determine mentally that 49, 81, 121, and 4936 are prime to one another.
10. Similarly for 429, 490, and 12,347.

The Euclidean or long division method may also be illustrated by a single example, the one already considered.

1. $\because$ the g.c.d. is contained in each number it $\not>$ 9801.

2. The g.c.d. $\neq$ 9801, $\because$ that is not a factor of 33,759.

3. $\because$ the g.c.d. is a factor of 9801 and 33,759, it is a factor of 4356. P. 16, th. 2

4. $\therefore$ the g.c.d. $\not>$ 4356 and it is 4356 if that is a factor of 9801, 33,759, and 121,968.

```
                  3
      9801)33759
           29403       2
            4356)9801
                 8712       4
                 1089)4356
                      4356
```

5. But the g.c.d. $\neq$ 4356 $\because$ that is not a factor of 9801.

6. $\because$ the g.c.d. is a factor of 4356 and 9801, it is a factor of 1089. P. 16, th. 2

7. $\therefore$ the g.c.d. $\not>$ 1089 and it is 1089 if that is a factor of 4356, 9801, 33,759, and 121,968.

```
              112
      1089)121968
           1089
            1306
            1089
             2178
             2178
```

8. $\because$ 1089 is a factor of 4356, it remains to find whether it is a factor of 9801, 33,759, and 121,968.

9. $\because$ 1089 is a factor of itself and of 4356, it is a factor of 9801. P. 16, th. 2

10. $\therefore$ it is a factor of 33,759. P. 16, th. 2

11. And 1089 is a factor of 121,968, by trial.

12. $\therefore$ 1089 is the greatest common divisor of the three numbers.

Exercises. 1. Find, by the Euclidean method, the g.c.d. of 6961 and 9976.

2. Also of 8673 and 23,989.

3. Also of 2827, 3341, and 11,565.

4. Also of 5187, 14,421, and 3249.

Abbreviations of the Euclidean form will readily suggest themselves. Only two will, however, be considered.

1. If a factor is common to two numbers it must be a factor of their greatest common divisor. Hence, if seen, it may be suppressed in order to shorten the work, and introduced in the result. For example, the factor 2 in the problem below.

2. If a factor of either number is not common to the other it cannot be a factor of their greatest common divisor. Hence, if seen, it may be suppressed in order to shorten the work. For example, the factor 3^2 in the problem below.

Required the g.c.d. of 33,282 and 73,874.

$$
\begin{array}{r|l}
2 & 33282 \\ \hline
3 & 16641 \\ \hline
3 & 5547 \\ \hline
 & 1849
\end{array}
\qquad
\begin{array}{r|l}
2 & 73874 \\ \hline
 & 36937
\end{array}
$$

$$
\begin{array}{rrrrr}
 & 19 & & & \\
1849) & 36937 & & & \\
 & 18447 & & 1 & \\
 & 1806) & 1849 & & 42 \\
 & & & 43) & 1806 \\
 & & & & 86
\end{array}
$$

$\therefore$ g.c.d. $= 2 \times 43 = 86.$

Since a composite factor, like 9, may not be common to two numbers and yet may contain a factor, as 3, which is common, only prime factors should be rejected.

Exercises. 1. Find, by the Euclidean method, suppressing factors whenever it is advantageous, the g.c.d. of 845,315 and 265,200.

2. Also of 4,010,401 and 4,011,203.
3. Also of 16,897 and 58,264.
4. Also of 40,033 and 129,645.
5. Also of 29,766 and 208,362.
6. Also of 376, 940, 1034, and 1081.
7. Reduce to lowest terms the fractions $\frac{1067}{1139}$ and $\frac{2077}{1943}$.
8. Also the fractions $\frac{1849}{2021}$ and $\frac{2921}{4699}$.
9. Also the fractions $\frac{5363}{7439}$ and $\frac{8021}{10489}$.
10. Also the fractions $\frac{9131}{10327}$ and $\frac{8843}{9731}$.

IV. Least Common Multiple.

In adding common fractions it is necessary to reduce them to fractions having a common denominator, and preferably to fractions having their least common denominator. Hence, when common fractions were largely used, this subject was of great importance. The extensive use of the decimal fraction at the present time has, however, rendered unnecessary any such elaborate treatment of the subject as the earlier works present. As in the case of the greatest common divisor, the interest is now in the theory rather than in the practical applications.

The method of factoring may be illustrated by the following example: Required the least common multiple of 9801, 33,759, and 121,968.

1. $9801 = 3^4 \times 11^2$.
2. $33{,}759 = 3^2 \times 11^2 \times 31$.
3. $121{,}968 = 3^2 \times 11^2 \times 2^4 \times 7$.
4. And since the least common multiple contains all three numbers, and no unnecessary factors, it contains the factors 3^4, 11^2, 2^4, 7, and 31 and no others, and therefore is 34,029,072.

The greatest common divisor may be used in finding the least common multiple. Thus, in the above example:

1. The g.c.d. of 9801, 33,759, and 121,968 is 1089.
2. $\therefore$ this factor enters once and no more into the l.c.m.
3. $\therefore$ this may be suppressed from any two of the numbers, as from 9801 and 121,968, leaving 9 and 112, and the other number may be multiplied by these two factors.
4. And since 1089 contains all the common factors, $9 \times 112 \times 33{,}759$, or 34,029,072, contains all the factors of the three numbers, without repetition, and is therefore their least common multiple.

Exercises. 1. What is meant by one number being a divisor of another? a common divisor of two or more numbers? the greatest common divisor of two or more numbers?

2. What is meant by one number being a multiple of another? a common multiple of two or more numbers? the least common multiple of two or more numbers?

3. State the relative advantages of the two methods given for finding the greatest common divisor of several numbers.

4. Similarly for finding the least common multiple of several numbers.

5. Explain what is meant by a prime number; by two numbers being prime to each other. Is 2.5 a prime number? Are 8 and 21 prime numbers? prime to each other?

6. Find the least common multiple of 100, 101, and 103.

7. Also of 100, 240, and 515.

8. Also of 376, 1034, and 1081.

9. Also of 173,376 and 171,072.

10. The l.c.m. of two numbers is 96 and one of the numbers is 6, what values may the other number have?

11. Find the g.c.d. and also the l.c.m. of 763 and 2071. How does the product of the g.c.d. and the l.c.m. compare with the product of the two numbers?

12. Similarly for 2033 and 8239.

13. Similarly for 8321 and 9577.

14. Prove that the product of the g.c.d. and the l.c.m. of any two numbers always equals the product of the numbers.

15. Find two pairs of numbers whose g.c.d. is 12 and whose sum is 96.

16. Of numbers below 100, what ones have with 360 the g.c.d. 4?

17. Find the g.c.d. and the l.c.m. of 144, 176, and 272; also of 161, 253, and 299.

18. One of two cog wheels which work together has 21 cogs, and the other 11; if a certain cog of one wheel rests on a certain cog of the other, after how many revolutions of the smaller will they be in the same relative position? after how many revolutions of the larger?

19. Three steamers arrive at a certain port, the first every Monday, the second every 10 days, the third every 12 days; they all arrive on Monday, May 1. When will (*a*) the first and second next arrive together? (*b*) the first and third? (*c*) the second and third? (*d*) all three?

CHAPTER III.

Common Fractions.

THE decimal point began to be used about the beginning of the seventeenth century, but it was over a hundred years before the new decimal fractions were extensively taught. During the transition period the old style fractions were so generally used that they were distinguished from the others by the term "vulgar" or "common" fractions, a name which still remains, although the decimal form is now more generally employed. The word *vulgar* then meant *common* and is still used in England in speaking of these fractions, although not generally so employed in America. It should, however, be remembered that the decimal fraction is only a special kind of a common fraction written in a special way. Thus, 0.25 is merely $\frac{25}{100}$, or $\frac{1}{4}$; that is, it always has a denominator 10^n, and this denominator is not expressed, but is indicated by the position of the decimal point.

The student is already familiar with the various operations involving fractions. He needs, however, to review the reasons included in these operations, — reasons imperfectly presented (if at all) in the primary school, or since forgotten.

A *fraction* is one or more of the equal parts of a unit.

Thus, the fraction $\frac{2}{3}$ is two of the three equal parts of one, the fraction 0.5 is five of the ten equal parts of one, and the fraction $\frac{17}{3}$ is seventeen of the three equal parts of one.

This definition is incomplete. It excludes such fractions as $\frac{2.52}{17.3}$, $\frac{\frac{3}{8}}{\frac{5}{7}}$, $\frac{-1}{7}$, $\frac{3}{-5}$, 0.131313·····, 3.14159·····, etc. But it includes decimal fractions, and common fractions of the form $\frac{a}{b}$, a and b being positive integers, and these are the ones practically used. More scientifically defined, a fraction is an expressed division; but the treatment of the subject under this definition is so abstract as to be better adapted to more advanced works.

The terms *numerator* (Latin, *numberer*, because it numbers the parts) and denominator (Latin, *namer*, because it names the parts) need no definition.

A fraction is called a *proper fraction* when the numerator is less than the denominator; otherwise an *improper fraction.*

Fundamental properties of fractions.

I. *An integer may be expressed as a fraction with any given denominator.*

Thus, to express a as a fraction with the denominator b.

1. $\because 1 = \frac{b}{b}$, that is, b bths.
2. $\therefore a = a \times b$ $b$ths, or ab $b$ths,
   $= \frac{ab}{b}$.

Hence, any integer may be considered a fraction with the denominator 1, thus broadening somewhat the original idea of a fraction.

II. *An improper fraction may be reduced to an integer, or an integer plus a proper fraction.*

Thus, to express $\frac{37}{7}$ as an integer plus a proper fraction.

1. As \$37 contains \$7 5 times, with a remainder of \$2,
2. so $\frac{37}{7}$ contains $\frac{7}{7}$ 5 times, with a remainder of $\frac{2}{7}$.
3. That is, in $\frac{37}{7}$ there are 5 units + $\frac{2}{7}$.

The proof, though applied to a particular fraction, is evidently general. The same result could have been obtained by dividing the numerator by the denominator.

III. *Multiplying or dividing the numerator of a fraction by any number multiplies or divides, respectively, the value of the fraction by that number.*

1. $\frac{a \times b}{c} = a \times \frac{b}{c}$, because there are a times as many $c$ths as before.

2. $\frac{b \div a}{c} = \frac{b}{c} \div a$, " " " $\frac{1}{a}$th " " " " " ".

IV. *Multiplying or dividing the denominator of a fraction by any number divides or multiplies, respectively, the value of the fraction by that number.*

1. $\frac{b}{a \times c} = \frac{1}{a}$th of $\frac{b}{c}$, because if the unit is divided into a times as many parts, each part (and hence the fraction) is only $\frac{1}{a}$th as large.

2. $\frac{b}{c \div a} = a \times \frac{b}{c}$, because if the unit is divided into $\frac{1}{a}$th as many parts, each part (and hence the fraction) is a times as large.

V. *The value of a fraction is not altered by multiplying or dividing both terms by the same number.*

Since $n \times \frac{a}{b} = \frac{na}{b}$, by III, and $n \times \frac{na}{nb} = \frac{na}{b}$, by IV,

$\therefore$ $n \times \frac{a}{b} = n \times \frac{na}{nb}$, or $\frac{a}{b} = \frac{na}{nb}$, by ax. 7.

Addition and subtraction of fractions.

I. *When they have a common denominator, as* $\frac{a}{d} \pm \frac{b}{d}$, *where* $a > b$.

As $\$a \pm \$b = \$(a \pm b)$, the unit \$1 being the same,

so $\frac{a}{d} \pm \frac{b}{d} = \frac{a \pm b}{d}$, the unit $\frac{1}{d}$ being the same.

II. *When they have not a common denominator, they may be reduced to fractions having a common denominator* (by fundamental property V), *and preferably to fractions having the least common denominator.*

Multiplication of fractions. To *multiply* one number by another is to perform that operation upon the first which being performed on unity produces the second.

Since the primitive notion of multiplication is the taking of a number a certain number of *times*, as 5 times \$2, and since it is meaningless to take a number *3 inches times*, multiplication is considered as an operation in which the multiplicand may be either abstract or concrete, but in which the multiplier is always abstract.

To illustrate the definition, $2 \times \$3 = \$6$; since 1 is added to itself to produce the multiplier, \$3 is added to itself to produce \$6.

Similarly in the case of $\frac{a}{b} \times \frac{c}{d}$. To produce the multiplier $\frac{a}{b}$ from 1, 1 must be divided into b parts, and a of those parts taken. So to produce the product from the multiplicand $\frac{c}{d}$, the fraction $\frac{c}{d}$ must be divided into b parts (each being $\frac{c}{bd}$, by IV), and a of those parts taken, giving $\frac{ac}{bd}$.

The symbols of multiplication, $\times$ and $\cdot$, are usually read "of" after a pure fraction, but "times" after an integer or a mixed number. Thus, $\frac{3}{8} \times \$5$ is read "$\frac{3}{8}$ of \$5," but $1\frac{1}{4} \times \$5$ is read "$1\frac{1}{4}$ times \$5," this being the abridged form for "once \$5 and $\frac{1}{4}$ of \$5." (See p. 42.)

Division of fractions. *Division* may be defined as the operation of finding one of two factors, the quotient, when the product and the other factor are given.

For example, $2 \times \$5 = \$10$, $\therefore \$10 \div \$5 = 2$, and $\$10 \div 2 = \$5$.

To divide $\frac{a}{b}$ by $\frac{c}{d}$ is, therefore, to find a quotient q, such that

$$\frac{a}{b} = q \times \frac{c}{d}.$$

$\therefore \frac{ad}{bc} = q$, by multiplying equals by d, and dividing by c, by axs. 6 and 7, and fundamental properties III and IV.

It therefore appears that the quotient equals the product of the dividend and the reciprocal of the divisor, and can, therefore, be obtained by multiplication. Hence the familiar rule, "Invert the divisor and multiply," a rule that is always valid for abstract divisors.

Complex fractions of the form $\frac{\frac{a}{b}}{\frac{c}{d}}$ may be considered as the equivalent of $\frac{a}{b} \div \frac{c}{d}$, and treated accordingly.

And since $n \times \frac{c}{d} \times \frac{\frac{a}{b}}{\frac{c}{d}} = n \times \frac{c}{d} \times \frac{a}{b} \div \frac{c}{d} = n \times \frac{a}{b} \times \frac{c}{d} \div \frac{c}{d} = n \times \frac{a}{b}$;

and $$n \times \frac{c}{d} \times \frac{n \times \frac{a}{b}}{n \times \frac{c}{d}} = n \times \frac{a}{b}, \text{ by def. of division};$$

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{n \times \frac{a}{b}}{n \times \frac{c}{d}}.$$

That is, fundamental property V applies to complex fractions as well as to simple fractions.

Practical suggestions as to the treatment of fractions.

I. *Make free use of fundamental property* V, *multiplying or dividing the terms by the same number.*

For example, the fraction $\frac{693}{1287}$ should have the factors 9 and 11 suppressed at once, the fraction reducing to $\frac{7}{13}$. In the case of the complex fraction $\frac{1\frac{7}{8}}{5\frac{1}{4}}$ the factor 8 should be introduced, the fraction reducing to $\frac{15}{42}$, or $\frac{5}{14}$, an operation much simpler than division.

II. *In reducing to lower terms, it is best to reject simple factors at once, without attempting to find the greatest common divisor by the long process.*

For example, in the case of the fraction $\frac{693}{1287}$ above considered.

III. *Feel free to use the common fraction or the decimal as may be the more convenient in the computation in hand.*

For example, it is better to multiply or divide by $\frac{1}{4}$ than by 0.25, and by $\frac{1}{8}$ than by 0.125. But 0.2 is an easier operator than $\frac{1}{5}$, and 0.04 is easier than $\frac{1}{25}$.

Exercises. 1. Explain the reduction of $5\frac{2}{7}$ to $\frac{37}{7}$.

2. Also the reduction of $\frac{61}{8}$ to $7\frac{5}{8}$.

3. Perform at sight the following multiplications: 0.125 of 640, $0.33\frac{1}{3}$ of 903, 0.25 of 500, and 0.5 of 720.

4. Also the following divisions: $840 \div 0.125$, $69 \div 0.33\frac{1}{3}$, $200 \div 0.25$, and $68 \div 0.5$.

5. Reduce to lowest terms the fractions $\frac{209}{380}$, $\frac{216}{3186}$, $\frac{7854}{2160}$, and $\frac{3300}{4235}$.

6. Simplify $\dfrac{3\frac{4}{17}}{1\frac{1}{34}}$, $\dfrac{\frac{4}{5}}{\frac{7}{8}}$, $\dfrac{0.2}{\frac{3}{7}}$, and $\dfrac{4}{\frac{2}{3}}$.

7. Add $\frac{7}{2431}$, $\frac{29}{2717}$, $\frac{589}{3553}$, $\frac{3446}{4199}$. (The factors of 4199 are 13, 17, and one other.)

8. Also the fractions $\frac{1001}{1859}$, $\frac{259}{481}$, and $\frac{1596}{1729}$.

9. $\frac{5}{12} + \frac{5}{12}$ of $\frac{4}{11} + \frac{5}{12}$ of $\frac{4}{11}$ of $\frac{3}{10} + \frac{5}{12}$ of $\frac{4}{11}$ of $\frac{3}{10}$ of $\frac{2}{9} + \frac{5}{12}$ of $\frac{4}{11}$ of $\frac{3}{10}$ of $\frac{2}{9}$ of $8 = ?$

10. $\frac{1}{6}$ of $\frac{4}{51}$ of $\frac{5}{24}$ of $\frac{3}{25} = ?$

11. $1\frac{1}{2} \times 2\frac{1}{3} \times 3\frac{1}{4} \times 4\frac{1}{5} \times 5\frac{1}{6} = ?$

12. Divide $\frac{5}{6}$ of $\frac{7}{8}$ of $3\frac{1}{5}$ by $\frac{2}{3}$ of $\frac{5}{6}$ of $4\frac{1}{5}$.

13. Divide $\frac{7}{20}$ of $\frac{5}{24}$ by $\frac{4}{25}$ of $\frac{5}{24}$.

14. Of the three fractions $\frac{11}{19}$, $\frac{18}{29}$, and $\frac{23}{38}$, which is greatest? which least?

15. Which is greater, $\frac{20}{17}$ or $\frac{25}{23}$? $\frac{4}{5}$ of $\frac{5}{6}$ or $\frac{3}{4}$ of $\frac{4}{5}$?

16. What is the effect of adding the same number to both terms of a proper fraction on the value of the fraction? Prove it for the general case of $\frac{a}{b}$, where $a < b$.

17. Investigate the same for the fraction $\frac{a}{b}$, when $a > b$.

18. Show that the fraction $\dfrac{2+4+6}{3+5+7}$ lies between the greatest and the least of the fractions $\frac{2}{3}$, $\frac{4}{5}$, $\frac{6}{7}$.

19. A "magic square" is a square array of numbers such that the sums of the numbers in the rows, columns, and two diagonals are equal, as in the annexed illustration. Insert the fractions to complete the following magic square:

4	3	8
9	5	1
2	7	6

$\frac{1}{10}$	$\frac{4}{5}$	*	$\frac{7}{6}$
$\frac{22}{15}$	*	*	$\frac{8}{15}$
*	$\frac{7}{15}$	*	$\frac{5}{6}$
$\frac{19}{15}$	$\frac{11}{10}$	*	$\frac{11}{15}$

CHAPTER IV.

Powers and Roots.

The cases are few in practical business where either involution or evolution is used. In scientific work, numbers often have to be raised to powers, and roots have to be extracted, but the operations are usually performed with the help of tables of powers, roots, or logarithms. The value of the subject may, therefore, be said to lie largely in the exercise of the reasoning powers. Hence, in the present chapter more attention is directed to the reasons for the various steps than to short methods of securing results.

I. Involution.

Symbolism. a^2 is read "a square," or "a to the second power," and means $a \cdot a$; a^3 is read "a cube," or "a to the third power," and means $a \cdot a \cdot a$; a^4 is read "a to the fourth power," and means $a \cdot a \cdot a \cdot a$; and, in general, a^n is read "a to the nth power," and means $a \cdot a \cdots\cdots$ (n times).

This symbolism and the notion of power have been extended, thus: $\because$ a^2 is obtained by dividing a^3 by a, so a^1 is defined as $\frac{a^2}{a}$ or a, and is read "a to the first power," and a^0 is defined as $\frac{a^1}{a}$ or 1, and is read "a to the zero

power," and a^{-1} is defined as $\frac{a^0}{a}$ or $\frac{1}{a}$, and is read "a to the minus first power," and a^{-2} is defined as $\frac{a^{-1}}{a}$ or $\frac{1}{a^2}$, and is read "a to the minus second power," and, in general, a^{-n} is defined as $\frac{1}{a^n}$, and is read "a to the minus nth power."

A further extension has also been made to include fractional powers, thus:

$\because a^2 = \sqrt{a^4}$, and $a^1 = \sqrt{a^2}$, so $a^{\frac{1}{2}}$ is defined to mean $\sqrt{a}$, and is read "a to the $\frac{1}{2}$ power" or "the square root of a"; and, in general, $a^{\frac{1}{n}}$ is defined as $\sqrt[n]{a}$, and $a^{\frac{m}{n}}$ as the mth power of the nth root of a. Thus, $a^{1.25}$ means the 125th power of the 100th root of a.

Raising numbers to high powers. It occasionally becomes necessary to raise a number like 2 to some high power, as in the case of 2^{30}. Here the computer should recall the fact that $a^m \cdot a^n = a^{m+n}$, and proceed as indicated in the annexed multiplication.

$$
\begin{array}{rr}
2^4 = & 16 \\
2^4 = & 16 \\
\hline
2^8 = & 256 \\
2^8 = & 256 \\
\hline
2^{16} = & 65536 \\
2^{16} = & 65536 \\
\hline
2^{32} = & 4294967296 \\
2^{30} = 2^{32} \div 2^2 = & 1073741824
\end{array}
$$

Powers of binomials. In the extraction of roots by the method to be considered it is necessary to know the corresponding powers of binomials. The student may expand the following:

$(f+n)^2 = f^2 + 2fn + n^2.$
$(f+n)^3 = f^3 + 3f^2n + 3fn^2 + n^3.$
$(f+n)^4 = (?)$. Obtain it by squaring $(f+n)^2$.
$(f+n)^5 = f^5 + 5f^4n + 10f^3n^2 + 10f^2n^3 + 5fn^4 + n^5.$
$(f+n)^6 = (?)$. Obtain it by squaring $(f+n)^3$.
$(f+n)^7 = (?)$.

Exercises. 1. Square 41 by using the formula for $(f+n)^2$.

2. Cube 22 by using the formula for $(f+n)^3$.

3. Similarly, find the values of 11^2, 11^3, 11^4, and from the results find the values of 11^5 and 11^7 by single multiplications.

4. Express 2^{-4}, 4^{-1}, 5^{-2}, and 10^{-5} as decimal fractions.

5. Express 0.04 and 0.03125 as negative powers of integers.

6. Prove that no number ending in 2, 3, 7, or 8 can be a perfect square.

7. Prove that the square of a number ending in 5 ends in 025, 225, or 625.

8. Prove that a square must end in 0, 1, 4, 5, 6, or 9.

9. Prove that a cube may end in any of the digits.

10. Prove that the cube of a number ending in 5 must end in 125, 375, 625, or 875.

11. Prove that the 5th power of a number ends in the same digit as the number itself.

12. If the student has taken the subject of imaginary numbers in algebra, but not otherwise, he may solve the following:

(*a*) $(-\frac{1}{2}+\frac{1}{2}\sqrt{-3})^2 = ?$ (*b*) $(-\frac{1}{2}-\frac{1}{2}\sqrt{-3})^2 = ?$

(*c*) $(-\frac{1}{2}+\frac{1}{2}\sqrt{-3})^3 = ?$ (*d*) $(-\frac{1}{2}-\frac{1}{2}\sqrt{-3})^3 = ?$

II. Square Root.

The *square root* of a perfect second power is one of the two equal factors of that power.

In case a number is easily factored, the square root may be found by this means, as explained on p. 37.

A number which is not a perfect second power has not two equal factors. It is, however, said to have a square root to any required degree of approximation. Thus, the square root of n to 0.1 is that number of tenths whose square differs from n by less than the square of any other number of tenths.

E.g., the square root of 2 to 0.1 is 1.4, to 0.01 is 1.41, etc.

It should be observed that under these definitions only abstract numbers can have square roots. Thus, 4 is the product of 2 and 2, hence $2=\sqrt{4}$; but no number multiplied by itself equals \$4, or 4 feet, or 4 square meters.

The general theory of this subject is best understood by following the solution of a problem. Suppose the square root of 547.56 be required.

Let $f=$ the *f*ound part of the root at any stage of the operation, and $n=$ the *n*ext digit to be found.

Then $$(f+n)^2=f^2+2fn+n^2.$$

	2 3. 4 5‘47.56	
The greatest square in 500 is	4 00.	that is, the square of 20, or f^2.
∵ 20 has been found, and 20^2, or f^2, subtracted, this 147.56 must contain $2fn+n^2$, or $2\cdot 20\cdot n+n^2$. ∴ by dividing by $2\cdot 20$, or 40, n can be found approximately. ∴ $n=3$.	1 47.56	contains $2fn+n^2$. f is *now* 20 because that is all that has been found. n is *now* 3, the next digit.
∴ $2fn+n^2$, or $2\cdot 20\cdot 3+3^2=$	1 29	$=2fn+n^2$.
∵ 23 has been found, and 23^2, or f^2 $(=400+129)$, subtracted, this 18.56 must contain $2fn+n^2$, or $2\cdot 23\cdot n+n^2$. ∴ by dividing by $2\cdot 23$, or 46, n can be found approximately. ∴ $n=0.4$. ∴ $2fn+n^2$,	18.56	contains $2fn+n^2$. f is *now* 23 because that has been found. n is *now* 0.4, the next digit.
or $2\cdot 23\cdot 0.4+0.4^2$ =	18.56	$=2fn+n^2$.

The actual computation may be conveniently arranged in either of the following ways, the first being preferable for the majority of students.

	2 3. 4
	$\sqrt{5‘47.56}$
	4 00
40	1 47
43	1 29
46	18.56
46.4	18.56

	2 3. 4
	$\sqrt{5‘47.56}$
43	1 47
46.4	18.56
	0

For those who desire a complete explanation of the process a more extended discussion appears on p. 36.

$$
\begin{array}{llrlll}
\multicolumn{3}{l}{f^2 + 2fn + n^2} & & & \\
 & & 2\ \ 3.\ \ 4 & & & \\
 & & 5{`}47.56 & & & \\
 & f_1^2 = & 4\ 00.00 & & & \\
\hline
2f_1 & = 40 & 1\ 47.56 & \text{contains} & 2f_1n_1 + n_1^2 & f_1 = 20 \\
2f_1 + n_1 & = 43 & 1\ 29.00 & = & \text{``} & n_1 = 3 \\
\hline
2f_2 & = 46 & 18.56 & \text{contains} & 2f_2n_2 + n_2^2 & f_2 = 23 \\
2f_2 + n_2 & = 46.4 & 18.56 & = & \text{``} & n_2 = 0.4 \\
\hline
\end{array}
$$

1. $\because$ the highest order of the power is 100's, the highest order of the root is 10's, and it is unnecessary to look below 100's for the square of 10's.

2. Similarly, it is unnecessary to look below 1's for the square of 1's, below 100ths for the square of 10ths, etc. [These places may be indicated by points (‘), as in the above example.]

3. The greatest square in the 100's is 400, which is the square of 20, which may be called f_1 (read "f-one"), the first found part of the root.

4. Subtracting, 147.56 contains $2fn + n^2$ because f^2 has been subtracted from $f^2 + 2fn + n^2$, where f stands always for the *found* part and n for the *next order* of the root.

5. $2fn + n^2$ is approximately the product of $2f$ and n, and hence, if divided by $2f$, the quotient is approximately n. $\therefore n = 3$.

6. $\therefore 2f + n = 2 \times 20 + 3 = 43$, and this, multiplied by n, equals $2fn + n^2$.

7. $\because f^2$ has already been subtracted, after subtracting $2fn + n^2$ there has been subtracted $f^2 + 2fn + n^2$, or $(f + n)^2$, or 23^2.

8. Calling 23 the second found part, f_2, and noticing that $f_2 = f_1 + n_1$, it appears that 23^2, or f_2^2, has been subtracted.

9. $\therefore$ the remainder 18.56 contains $2f_2n_2 + n_2^2$.

10. Dividing by $2f_2$ for the reason already given, $n_2 = 0.4$.

11. $\therefore 2f_2 + n_2 = 46.4$, and $18.56 = 2f_2n_2 + n_2^2$, as before.

12. Similarly, the explanation repeats itself after each subtraction.

13. Students will remember from algebra that every number has two square roots, one + and the other −. $\therefore \sqrt{547.56} = \pm 23.4$, but the positive root is the only one likely to be needed in practice.

The subject of square root is still further discussed in the Appendix, Note 1.

Common fractions. There are three general methods for extracting the square root of a common fraction.

1. The square root of both terms may be extracted, as is advisable when each is a square number.

$$E.g.,\quad \sqrt{\tfrac{25}{36}} = \pm\tfrac{5}{6}.$$

2. The fraction may be reduced to the decimal form, as is advisable when this can easily be done.

$$E.g.,\quad \sqrt{\tfrac{1}{5}} = \sqrt{0.2} = 0.447 \cdots\cdots$$

3. The fraction may be reduced to an equal fraction whose denominator is a square number.

E.g., $\sqrt{\tfrac{2}{7}} = \sqrt{\tfrac{14}{49}} = \tfrac{1}{7}\sqrt{14} = \tfrac{1}{7}$ of $\pm 3.741657 = \pm 0.534522$, an easier method in most cases than either of the two just mentioned.

Factoring method. The cube root of a perfect third power is one of the three equal factors of that power, and similarly for the fourth, fifth, ····· nth roots. As mentioned on p. 34, such roots can often be found by factoring.

E.g., $85,766,121 = 3\cdot3\cdot3\cdot3\cdot3\cdot3\cdot7\cdot7\cdot7\cdot7\cdot7\cdot7;$

$\therefore \sqrt{85,766,121} = 3\cdot3\cdot3\cdot7\cdot7\cdot7 = 9261,$

$\sqrt[3]{85,766,121} = 3\cdot3\cdot7\cdot7 = 441,$

and $\sqrt[6]{85,766,121} = 3\cdot7 = 21.$

Even in case a number is not a perfect power the factoring method can often be advantageously used.

$$E.g.,\quad \sqrt{882} = \sqrt{2\cdot3^2\cdot7^2} = 3\cdot7\sqrt{2} = 21\sqrt{2}$$
$$= 21\cdot1.4142 \cdots\cdots = 29.698 \cdots\cdots$$

Exercises. 1. The student may test his knowledge of the general theory by answering the following questions: (*a*) If you separate into periods of two figures each, where do you begin? Consider, for example, the square root of 14.4.

(*b*) Why does the remainder contain $2fn + n^2$ the first time? the second?

(*c*) Why is $2f$ always taken as the *trial divisor*?

(*d*) Why is n added to $2f$ to make the *complete divisor*?

(*e*) In the example on p. 35, why does 129 equal $2fn + n^2$?

(*f*) In that example, how can 129 and 18.56 each equal $2fn + n^2$?

2. Extract the square roots of the following numbers, writing out the solutions in the full form given on p. 35:

(*a*) 80.4609. (*b*) 8226.49. (*c*) 1280.9241. (*d*) 0.21224449.
(*e*) 12.8881. (*f*) 0.49112064. (*g*) 592330.3369. (*h*) 32.26694416.

3. Extract the square roots of the following numbers, abridging the solution as in the first at the foot of p. 35:

(*a*) 40509.6129. (*b*) 0.501361708761. (*c*) 234.579856.
(*d*) 96.27534400. (*e*) 1.47403881. (*f*) 416.05800625.
(*g*) 28597039.6644. (*h*) 8260.628544. (*i*) 85747600.

4. Extract the square roots, to 0.001, of the following numbers:

(*a*) 0.0068. (*b*) 20. (*c*) 2. (*d*) 951.
(*e*) 680. (*f*) 809. (*g*) 13. (*h*) 1000.

5. Extract the square roots, to 0.00001, of the following numbers:

(*a*) 976. (*b*) 887. (*c*) 0.565. (*d*) 3.

6. Decide which of the three methods for extracting the square root of a common fraction is the best for each of the following numbers, giving the reason, and extract the root accordingly, to 0.001.

(*a*) $\frac{5}{17}$. (*b*) $\frac{1}{53}$. (*c*) $\frac{2}{3}$. (*d*) $\frac{5476}{9604}$.
(*e*) $\frac{3}{57}$. (*f*) $\frac{1521}{8281}$. (*g*) $\frac{4}{5}$. (*h*) $\frac{11}{67}$.
(*i*) $\frac{25}{2304}$. (*j*) $\frac{1}{625}$. (*k*) 50^{-2}. (*l*) 25^{-6}.

7. By separating into factors, find the square roots of 2304, 9216, 396,900, 194,481, 11,025, 117,649.

8. Similarly, the cube roots of 46,656, 91,125, 1,953,125, 11,390,625, 250,047,000, 85,766,121, 1,771,561.

9. Similarly, the fourth roots of 15,752,961, 43,046,721, 59,969,536, 96,059,601.

10. Similarly, the fifth roots of 59,049, 4,084,101, 9,765,625, 3,486,784,401.

11. Similarly, the sixth roots of 34,012,224, 113,379,904, 177,978,-515,625.

12. The following sums are the squares of what numbers?

(*a*) $11^2 + 60^2$. (*b*) $8088^2 + 1,022,105^2$. (*c*) $13,552^2 + 936,975^2$.
(*d*) $18^2 + 19^2 + 20^2 + 21^2 + 22^2 + 23^2 + 24^2 + 25^2 + 26^2 + 27^2 + 28^2$.

13. Draw a square whose side is $f + n$ (f may be taken as $\frac{3}{4}$ of an inch and n as $\frac{1}{4}$ of an inch). From this figure, show that the square on $f + n$ is made up of the square on f, plus the square on n, plus two rectangles which are f long and n wide, thus illustrating the fact that

$$(f + n)^2 = f^2 + 2fn + n^2.$$

14. From the figure of Ex. 13, show that after f^2 is taken away there remains $2fn + n^2$.

III. Cube Root.

The complete definition of cube root may be inferred from that of square root. Suppose the cube root of 139,798,359 be required. Since the theory so closely resembles that of square root, the explanation is given in analogous form.

$f^3 + 3f^2n + 3fn^2 + n^3.$

$$\begin{array}{r} 5\quad 1\quad 9 \\ 139{,}798{,}359 \\ f^3 = 125{,}000{,}000 \end{array}$$

$3f^2$	$3fn + n^2$	$3f^2 + 3fn + n^2$	
			14,798,359 contains $3f^2n + 3fn^2 + n^3$. $f_1 = 500$.
750,000	15,100	765,100	$7{,}651{,}000 = 3f^2n + 3fn^2 + n^3$. $n_1 = 10$.
780,300	13,851	794,151	7,147,359 contains $3f^2n + 3fn^2 + n^3$. $f_2 = 510$. $7{,}147{,}359 = 3f^2n + 3fn^2 + n^3$. $n_2 = 9$.

1. $\because$ the highest order of the power is hundred-millions, the highest order of the root is 100's (why ?), and it is unnecessary to look below millions for the cube of 100's. (Why ?)

2. Similarly, it is unnecessary to look below 1000's for the cube of 10's, below 1's for the cube of 1's, etc. (These periods may be indicated by points as in square root, if desired.)

3. The greatest cube in the hundred-millions is 125,000,000, the cube of 500. $\therefore$ 500 may be called f.

4. Subtracting, 14,798,359 contains $3f^2n + 3fn^2 + n^3$. (Why ?)

5. This is approximately the product of $3f^2$ and n, and hence if divided by $3f^2$ the quotient is approximately n. $\therefore n = 10$.

6. $\therefore 3fn + n^2 = 15{,}100$, and $3f^2 + 3fn + n^2 = 765{,}100$, and this, multiplied by n, equals $3f^2n + 3fn^2 + n^3$.

7. $\because f^3$ has already been subtracted, after subtracting $3f^2n + 3fn^2 + n^3$ there has been subtracted $(f + n)^3$, or 510^3.

8. Calling 510 the second found part, f_2, it appears that f_2^3 has been subtracted. $\therefore$ the remainder contains $3f^2n + 3fn^2 + n^3$.

9. The explanation now repeats itself as in square root.

In practice, the work is usually arranged somewhat as follows:

	5 1 9
	139,798,359
	125
7500	14 798
7651	7 651
780300	7 147 359
794151	7 147 359

The subject of cube root is still further discussed in the Appendix, Note II.

Exercises. 1. (*a*) If you separate into periods of three figures each, why do you do so? Where do you begin? Why? Consider, for example, the cube root of 13.31.

(*b*) Why does the second remainder contain $3f^2n + 3fn^2 + n^3$?

(*c*) Why is $3f^2$ always taken as the trial divisor?

(*d*) Why is $3fn + n^2$ added to $3f^2$ to make the complete divisor?

(*e*) In the example on p. 39, why does 7,651,000 equal $3f^2n + 3fn^2 + n^3$?

(*f*) In that example, how can 7,651,000 and 7,147,359 each equal $3f^2n + 3fn^2 + n^3$?

2. Extract the cube roots of the following numbers, writing out the solutions in the full form given on p. 39:

(*a*) 139,798,359. (*b*) 248,858.189. (*c*) 0.004657463.
(*d*) 19.902511. (*e*) 0.000091733851. (*f*) 731.432701.

3. Extract the cube roots of the following numbers, abridging the solution as suggested in square root and at the top of this page:

(*a*) 553,387,661. (*b*) 381.078125. (*c*) 997.002999.
(*d*) 0.051064811. (*e*) 0.0001851930. (*f*) 0.876467493.

4. Extract the cube roots, to 0.001, of the following numbers:

(*a*) 251. (*b*) 455,000. (*c*) 0.57. (*d*) 0.27.
(*e*) 998. (*f*) 0.007. (*g*) 0.194104601. (*h*) 0.47637955.

5. Explain three methods of extracting the cube root of a common fraction, analogous to those given for square root.

6. Decide which of these three methods is the best for each of the following numbers, giving the reason, and extract the root accordingly, to 0.001:

(*a*) $\frac{2}{3}$. (*b*) $\frac{1}{7}$. (*c*) $\frac{2}{5}$. (*d*) $\frac{3}{11}$.

CHAPTER V.

The Formal Solution of Problems.

The most important portion of arithmetic, considered from the business standpoint, has already been completed, especially essential being that part which treats of addition, subtraction, multiplication, and division of integers and fractions. The subsequent portions of the subject are taught for the business and scientific principles involved, but largely as an exercise in logic. And since the operations mentioned have been the subject of extensive drill in the lower grades, it is neither necessary nor advisable to preserve them, after checking the result of each computation, in the treatment of applied problems. The solution should now be logically arranged in steps numbered for reference, the complete operations being preserved whenever the teacher advises. In this way, the logic of the solution stands out prominently, while on the other hand there is no loss in the way of arithmetical computations.

At every stage of the solution time and energy should be economized by resort to factoring and cancellation. It is a good rule, — *never multiply till you have to, always factor if you can.* The advantages of this rule are seen in the problem solved on p. 47.

The student should also be advised as to the proper use of symbols and language, and to this end a few suggestions may be of value.

I. Symbols.

The common symbols for multiplication are $\times$ and $\cdot$, the latter being preferable for students sufficiently mature not to confuse it with the decimal point. It is advisable to write the multiplier first because (*a*) it is usually read first, (*b*) the tendency among leading writers is to place it first, and (*c*) in an algebraic expression like $4x$ the first factor is usually looked upon as the multiplier.

Thus, if 1 book costs \$2, 123 books, at the same rate, will cost $123 \cdot \$2$. This would be indicated by the step $123 \cdot \$2 = \$246$, but the actual multiplication would of course be $2 \cdot 123$, on the principle that $123 \cdot 2 \cdot \$1 = 2 \cdot 123 \cdot \$1$.

The symbols may therefore be read as follows:

$2 \cdot \$3$, or $2 \times \$3$, "2 times \$3," or "2 into \$3";
$\$3 \cdot 2$, or $\$3 \times 2$, "\$3 multiplied by 2."

The word "times" in this connection has a much broader meaning than that assigned when arithmetic was in its infancy. Thus, we say "$2\frac{1}{2}$ times \$4," meaning thereby "2 times \$4 and $\frac{1}{2}$ of \$4." It is not customary, however, to use the word after a proper fraction; thus, $\frac{2}{3} \cdot \$4$ is read "$\frac{2}{3}$ *of* \$4," and $\frac{2}{3}$ ft. is read "$\frac{2}{3}$ of a foot." Hence, $2\frac{1}{2}$ *times* \$4 has acquired a meaning; but to look out of the window $2\frac{1}{2}$ *times* is nonsense.

The general agreements of mathematicians as to the relative weight of symbols should also be understood. The usage varies in different countries, however, and occasionally is not entirely settled in any one. The following may be taken as indicating the rules followed by the leading writers of the day.

The absence of a sign between two letters, and the fractional notation, indicate operations to be performed before any others.

Thus, in the expression $a \div cd \div \frac{e}{g}$, the multiplication cd is first performed; then the division of e by g; then the other divisions in order, beginning at the left.

The word "of" following a fraction stands next as to weight.

Thus, in $a \div \frac{1}{2}$ of cd, the multiplication cd is first performed; then the multiplication by $\frac{1}{2}$; then the division of a by the result.

The symbols $\cdot$, $\times$, $\div$, / stand next, one having the same weight as another.

Thus, in $1 \cdot 3 \times 4 \div 2 \times 6/3$, the operations are performed in order from left to right, the result being 12.

The symbols $+$, $-$ stand next, one having the same weight as the other.

Thus, $2 + 3 \cdot 6 - 5 + 4 \cdot 8 \cdot \frac{5}{8} = 35$.

The symbol :, when used as a symbol of ratio, stands next.

Thus, $2 + 3 : 4 + 1 = 5 : 5 = 1$. But since the symbol is one of division, and in some countries is the leading symbol of that operation, it is frequently given the same weight as the $\div$. In that case, $2 + 3 : 4 + 1 = 3\frac{3}{4}$, while $(2 + 3) : (4 + 1) = 1$.

The other common symbols are sufficiently understood already, or are explained elsewhere in this work.

Historical Note. The symbols $+$ and $-$ were used by Widmann in an arithmetic published at Leipzig in 1489, $=$ by Recorde in an algebra published in 1557, $\times$ by Oughtred in 1631, the dot ($\cdot$) as a symbol of multiplication by Harriot in 1631, the absence of a sign between two letters to indicate multiplication by Stifel in 1544, : as a symbol of division by Leibnitz, $\div$ as a symbol of division by Rahn in an algebra published at Zurich in 1659, $>$ and $<$ by Harriot in 1631. The symbols $\neq$, $\not>$, $\not<$, indicating "not equal," etc., are recent. Parentheses were first used as symbols of aggregation by Girard in 1629. The decimal point came into use in the seventeenth century; it seems to have appeared first in a work published by Pitiscus in 1612, but it was not extensively employed until more than a century later. Positive integral exponents in the present form were first used by Chuquet in 1484. The symbol $\sqrt{}$ was first used in this form by Rudolff in 1525.

II. Language.

The student should also guard against statements like the following: "2 times greater than 3" for "2 times as great as 3"; "3 + 1 equals *to* 4"; "\$4 + 3" for "\$4 + \$3"; "2 × 3 = \$6" for "2 × \$3 = \$6"; "2 is contained in \$6 \$3 times" for "\$6 divided by 2 equals \$3," or "$\frac{1}{2}$ of \$6 equals \$3."

III. Methods.

There is no general method of solution covering all problems; if there were, the subject would lose substantially all of its value as an exercise in logic. The student should feel encouraged to put into the work all the individuality possible, only being sure (1) that each statement is true, (2) that each result is checked, (3) that his solution involves no undue labor.

1. **Analysis in general.** The solution of any problem of applied arithmetic requires analysis of some kind; in other words, the application of a student's common sense. Two types are here given, and it will be seen that if the steps are properly arranged the oral analysis is a simple matter, beginning at each stage with a "since" and reasoning to a "therefore."

Problem. If the average velocity of a bullet in going from a gun to a target is 1342 ft. per sec., and that of sound is 1122 ft. per sec., how much time will elapse, on a range of 1000 yds., between the time the bullet strikes the target and the time that the sound of the discharge reaches the target?

Solution. 1. $1000 \cdot 3$ ft. = 3000 ft.

2. The bullet goes 1 ft. in $\frac{1}{1342}$ sec.
3. ∴ it goes 3000 ft. in $\frac{3000}{1342}$ sec., or 2.24 secs.
4. Similarly, sound requires $\frac{3000}{1122}$ sec., or 2.67 secs.
5. 2.67 secs. − 2.24 secs. = 0.43 sec.

Analysis. $\because$ 1 yd. $\doteq$ 3 ft., $\therefore$ 1000 yds. $=$ 1000 $\cdot$ 3 ft.

$\because$ the bullet goes 1342 ft. in 1 sec., $\therefore$ it goes 1 ft. in $\frac{1}{1342}$ of 1 sec., and 3000 ft. in $3000 \cdot \frac{1}{1342}$ of 1 sec.

Similarly, sound goes 3000 ft. in $3000 \cdot \frac{1}{1122}$ of 1 sec.

The difference in time is evidently the result required.

Problem. At what time between 1 and 2 o'clock are the hands of a clock at right angles to each other?

Analysis. (The student should first draw the figure.)

$\because$ they are together at 12 it is readily seen that the minute hand must gain 60 minute-spaces on the hour hand to bring them together again.

$\therefore$ it must gain $(60 + 15)$ minute-spaces or else $(60 + 45)$ minute-spaces to bring them at right angles between 1 and 2.

$\because$ at 1 o'clock the minute hand is at 12 and the hour hand at 1.

$\therefore$ the former gains 55 minute-spaces in 1 hr., or 1 minute-space in $\frac{1}{55}$ hr.

$\therefore$ to gain 75 or 105 minute-spaces it requires $75 \cdot \frac{1}{55}$ hr. $=$ 1 hr. $21\frac{9}{11}$ mins., or $105 \cdot \frac{1}{55}$ hr. $=$ 1 hr. $54\frac{6}{11}$ mins.

Exercises. 1. What is the speed in feet per sec. of a train moving uniformly at the rate of 20 mi. per hr.? 60 mi. per hr.?

2. The earth's center moves about the sun at the average rate of 101,090 ft. per sec.; how many miles per hr.?

3. An elastic ball rebounds to a height which is $\frac{2}{9}$ of that through which it fell; on the third rebound it rises to a height of $\frac{4}{11}$ ft.; from what height did it first fall?

4. Each of the two arctic zones covers 0.02 of the earth's surface, and each of the temperate zones 0.26; what part is covered by the two torrid zones together?

5. A locomotive consumes $\frac{1}{60}$ of its tankful of water every mile; it starts with only $\frac{3}{5}$ of a tankful; how many miles has it gone when it has $\frac{2}{15}$ of a tankful left?

6. A steam engine using 28.5 tons of coal in 30 working days has an improvement effected rendering it necessary to use only 4.8 tons a week of 6 working days; how much is saved in a year of 300 working days, coal costing \$5.60 a ton, not considering the cost of the improvement?

7. If the pressure of air on the surface of a lake is 15 lbs. per sq. in., and if 1 cu. ft. of water weighs 1000 oz., find the pressure per sq. ft. at the depth of 100 ft.

8. The average daily motion of the earth about the sun is $59' \, 8.3''$ a day; that of Mars is $31' \, 26.5''$; if they moved in the same plane and kept these rates, how many days would elapse from the time they were in the same straight line on the same side of the sun to the time when the earth was again directly between Mars and the sun? Draw a diagram illustrating the problem.

9. Similarly for earth and Mercury, the latter's daily rate being $4° \, 5' \, 32.5''$.

10. Similarly for earth and Venus, the latter's daily rate being $1° \, 36' \, 7.7''$.

11. Similarly for earth and Neptune, the latter's daily rate being $21.5''$.

12. Similarly for Mercury and Venus (see Exs. 9, 10).

13. Similarly for Venus and Neptune (see Exs. 10, 11).

14. What are the relative positions of the earth, the moon, and the sun at the time of a new moon? The average daily motion of the moon about the earth is 13.1764°; the apparent daily motion of the sun in the same direction is 0.98565°; required the time from one new moon to another.

15. It is estimated that a cannon-ball leaving the earth at the rate of 500 mi. per hr., and continuing that rate to the nearest fixed star, would require about 4,500,000 yrs. for the journey. (*a*) Express in index notation, giving only the first two significant figures, the distance to the star. (*b*) Knowing that light travels about 186,000 mi. per sec., find, to 0.1, the number of years required for the light of the star to reach the earth. (Take $365\frac{1}{4}$ da. = 1 yr.)

16. Sound travels 65,400 ft. per min.; how far away is a gun whose report is heard 15 secs. after firing?

17. The average cost per day for tuition in the common schools of the United States is 8.2 cts. Estimating the number of pupils at 14,000,000, what is the total cost per day? What is the cost per year of 150 school days?

18. The velocity of light being 186,330 mi. per sec., how long does it take the light from the sun to reach the earth, the distance being 93,165,000 mi.?

19. A factory is insured for $2500 in one company, $3500 in another, and $2000 in another; it is damaged by fire to the extent of $4875; what portion of the loss should each company bear?

20. A man left by will to four persons the sums of $1000, $950, $800, $750, respectively; his estate produced only $2900; how much should each legatee receive?

2. **Unitary analysis** is so called because the student analyzes the problem by passing to one or more units. The method is very advantageous in the solution of many problems which, although of no especial value in business or in science, are still found in most text-books. While such problems are foreign to the spirit of the present work, a few are given by way of illustration. In all these cases the words "at the same rate" are to be understood.

Problem. How many pumps working 12 hrs. per da. will be required to raise 7560 tons of water in 14 da., if 15 pumps working 8 hrs. per da. can raise 1260 tons in half that number of days?

Solution. 1. 1260 t. raised in 7 da. of 8 hrs. each require 15 times the work of 1 pump.

2. 1 t. in 7 da. of 8 hrs. requires $\frac{15}{1260}$ times the work of 1 pump.
3. 1 t. in 1 da. of 8 hrs. " $7 \cdot \frac{15}{1260}$ "
4. 1 t. in 1 da. of 1 hr. " $8 \cdot 7 \cdot \frac{15}{1260}$ "
5. 7560 t. in 1 da. of 1 hr. " $7560 \cdot 8 \cdot 7 \cdot \frac{15}{1260}$ "
6. 7560 t. in 14 da. of 1 hr. " $\frac{1}{14} \cdot 7560 \cdot 8 \cdot 7 \cdot \frac{15}{1260}$ "
7. 7560 t. raised in 14 da. of 12 hrs. each require

$$\frac{1}{12} \cdot \frac{1}{14} \cdot 7560 \cdot 8 \cdot 7 \cdot \frac{15}{1260} \text{ times the work of 1 pump}$$
$$= \text{the work of 30 pumps.}$$

In actual practice it would be better to pass from step 1 to step 4, and then directly to step 7. It would be a waste of energy to perform the operations at each step; by waiting until the last step numerous cancellations simplify the computation.

Exercises. 1. If 10 yds. of cloth $\frac{3}{4}$ yd. wide cost \$6.25, how much will 15 yds. of that cloth 1 yd. wide cost at the same rate per sq. yd.?

2. How long will it take 12 men to do a piece of work which 8 men can do in 54 da.?

3. If a 5-ct. loaf of bread weighs 1.5 lbs. when wheat is 75 cts. per bu., what should it weigh when wheat is \$1.00 per bu.?

4. If 5 compositors in 16 da. of 10 hrs. each can set up 20 sheets of 24 pages each, 40 lines to a page and 40 letters to a line, in how many days of 8 hrs. each can 10 compositors set up a volume composed of 40 sheets of 16 pages to the sheet, 60 lines to a page and 50 letters to a line?

3. **The simple equation.** Few mathematicians now assert that the distinction between arithmetic and algebra lies in the use of letters as symbols of quantity. It is impossible to study exhaustively the science of number without using literal notation. The simple equation is now used in all of the grades of many grammar schools, and it is no innovation to suggest its use in a work of this nature.

No more difficult equation is necessary than one of the following type:

1. Given $ax + b = c$, to find x.
2. $ax = c - b$, by subtracting b from these equals. Ax. 3
3. $x = \frac{c - b}{a}$, by dividing these equals by a. Ax. 7
4. Check: Putting $\frac{c - b}{a}$ for x in step 1, $a \cdot \frac{c - b}{a} + b = c$.

A single illustration may be given:

What sum gaining $0.06\frac{1}{4}$ of itself in a year amounts to \$157.50 in 2 yrs.?

1. Let $x =$ the sum.
2. $2 \cdot 0.06\frac{1}{4} = 0.12\frac{1}{2}$.
3. $\therefore x + 0.12\frac{1}{2}x = \$157.50$.
4. $\therefore 1.12\frac{1}{2}x = \$157.50$.
5. $\therefore x = \$140$.

Exercises. 1. What number is that which divided by 17 equals 2.1?

2. Divide 10 into two parts such that twice one part equals 3 times the other.

3. A fulcrum is to be placed under a 3-ft. lever so as to divide it into two parts such that 1.2 times the first shall equal 4.8 times the second; how far is it from either end?

4. Alcohol as received in the laboratory is 0.95 pure; how much water must be added to a gallon of this alcohol so that the mixture shall be half pure?

5. Air is composed of 21 volumes of oxygen and 79 volumes of nitrogen; if the oxygen is 1.1026 times as heavy as air, the nitrogen is what part as heavy as air?

IV. CHECKS.

A good computer checks his work at every step, and the student who does this has no need of the printed answers to problems involving only numerical calculations.

Checks have already been given for the fundamental operations, the most valuable one being that of casting out nines. One other deserves especial mention in this connection : *Always form a rough estimate of the answer before beginning a solution.* Reach as close an approximation as possible in a short time. This will be found to check most large errors and to do away with the absurd results often given by careless students.

E.g., if the interest on \$475 for 1 yr. at $4\frac{1}{2}\%$ is required, the student should at once think that it is a little less than half the interest on \$1000, that is, a little less than half of \$45; he might therefore make the estimate \$20. The interest is really \$21.38.

In the following exercises form a rough estimate of the answers and *write down* the approximations. *Then* solve.

Exercises. 1. At $12\frac{1}{2}$ cts. a pound, how much will $2\frac{3}{4}$ lbs. of cheese cost? (In solving note that $12\frac{1}{2} = \frac{100}{8}$.)

2. At $37\frac{1}{2}$ cts. a yard, how much will $13\frac{1}{2}$ yds. of cloth cost? (In solving note that $37\frac{1}{2} = \frac{3}{8}$ of 100.)

3. At \$3.50 a barrel, how much will 68 bbls. of flour cost?

4. At \$1.70 a barrel, how much will 126 bbls. of apples cost?

5. At 45 cts. a yard, how many yards of cloth can be bought for \$6.75?

6. If a person's taxes are 5.8 mills on \$1, how much will they be on \$8500?

7. If 41 qts. of water weigh as much as 51 qts. of alcohol, and 1 qt. of water weighs 2.2 lbs., how much will 1 qt. of alcohol weigh?

8. Bronze contains by weight 91 parts of copper, 6 of zinc, and 3 of tin; how many pounds of each in 700 lbs. of bronze?

9. In drawing a picture of a tower which is 160 ft. high and 35 ft. in diameter, the diameter is to be represented by 5 in.; by how many inches should the height be represented?

Exercises. 1. How much water must be added to a 5% solution of a certain medicine to reduce it to a 1% solution?

2. If sound travels 5450 ft. in 5 secs. when the temperature is 32°, and if the velocity increases 1 ft. per sec. for every degree that the temperature increases above 32°, how far does sound travel in 8 secs. when the temperature is 70°?

3. $84\frac{1}{4}$ qts. of water are drawn through a pipe every $4\frac{1}{2}$ mins. from a tank containing 237 qts.; how many minutes will it take to empty the tank, supposing the water to continue to run at the same rate?

4. The total debt of the United States government Jan. 1, 1897, was about $1,785,412,641, and the estimated population on that day was 74,036,761; what was the debt per capita?

5. A clock is set on Monday at 7 A.M.; on Tuesday at 1 P.M., correct time, it is 3 mins. slow; how many minutes will it be behind at 7 A.M., correct time, on Saturday?

6. If a railroad charges $18 for transporting 12,000 lbs. of goods 360 mi., how much ought to be charged for transporting 15,000 lbs. of goods 280 mi. at the same rate?

7. If $\frac{3}{8}$ in. on a map corresponds to 7 mi. of a country, what distance on the map represents 20 mi.?

8. How much pure alcohol must be added to a mixture of $\frac{4}{5}$ alcohol and $\frac{1}{5}$ water, so that $\frac{9}{10}$ of the mixture shall be pure alcohol?

9. If 40 pupils use 6 boxes of crayons, 200 in a box, in 3 mo., how many boxes, 150 in a box, will be required, at the same rate, to supply 75 pupils for 2 mo.?

10. If the velocity of electricity is 288,000 mi. per sec., how long will it take electricity to travel around the earth, 24,900 mi.?

11. The respective rates per sec. at which sound travels through air, water, and earth are approximately 1130 ft., 4700 ft., and 7000 ft.; at these rates, in what time could sound be transmitted a distance of 6 mi. through each of these media? (1 mi. = 5280 ft.)

12. A certain sum of money gains $\frac{5}{8}$ of itself, the total amount then being $728; what is the sum gained?

13. One of the trains on the Caledonian railway from Carlisle to Stirling, $117\frac{3}{4}$ mi., makes the run in 124 mins.; the "Empire State Express" makes the run from Syracuse to Rochester, 80 mi., in 84 mins.; what is the average rate of each per hr.?

14. The total debt of the various states and territories at the time of the eleventh census was $1,135,210,442, which was $18.12 per capita; compute the total population at that time, correct to 1000.

CHAPTER VI.

Measures.

THE earlier business arithmetics contained a large number of tables of measures, a necessity when the world was divided into relatively small states, each with its own system of coinage, weights, etc. As an example of the number of tables in use in a single country, there were nearly four hundred ways of measuring land in France at the close of the eighteenth century. Moreover, certain trades adopted special measures, thus adding to the confusion. The result is seen in the tables of Troy, avoirdupois and apothecary weights, the wine, beer, apothecary and common measures of capacity, besides numerous special units practically obsolete in general business in America, as the stone, long hundredweight, etc.

For common use to-day, only a few tables are needed. If one is to enter some trade which continues to use special measures, as that of druggist, the tables should be learned at that time as part of his technical education. Similarly, in an exchange office one must learn a considerable number of money systems, but for general information three or four suffice. The problems here set for review require only those tables in general use in business or in the sciences.

The tables on pp. 52 and 53 are inserted chiefly for reference. They include those which the student will most need to review. The metric tables are given on pp. 61 and 62.

Tables of Common Measure Needing Review.

Counting by 12.

1 dozen (doz.)	$= 12$.
1 gross (gro.)	$= 12^2$.
1 great gross (gt. gro.)	$= 12^3$.

Counting Sheets of Paper.

24 sheets	= 1 quire.
20 quires, or 480 sheets	= 1 ream.

Common Measures of Length.

12 inches (in.)	= 1 foot (ft.).
3 feet	= 1 yard (yd.).
$5\frac{1}{2}$ yards, or $16\frac{1}{2}$ ft.	= 1 rod (rd.).
320 rods, or 5280 ft.	= 1 mile (mi.).

Surveyors' Measures of Length.

7.92 inches	= 1 link (li.).
100 links	= 1 chain (ch.).
80 chains	= 1 mile.

Miscellaneous Measures of Length.

4 inches	= 1 hand.
6 feet	= 1 fathom.
1.15 miles, nearly,	= 1 knot, or 1 nautical or geographical mile.

Liquid Measure. Capacity.

4 gills (gi.)	= 1 pint (pt.).
2 pints	= 1 quart (qt.).
4 quarts	= 1 gallon (gal.)
	= 231 cubic inches.

Barrels and hogsheads vary in size.

Dry Measure. Capacity.

2 pints	= 1 quart.
8 quarts	= 1 peck (pk.).
4 pecks	= 1 bushel (bu.)
	= 2150.42 cu. in.

Avoirdupois Weight.

16 ounces (oz.)	= 1 pound (lb.).
100 pounds	= 1 hundredweight (cwt.).
2000 pounds	= 1 ton (t.).

2240 pounds = 1 long ton, little used in America except in wholesale transactions in mining products, and not generally there.

Common Measures of Surface.

144 square inches (sq. in.)	= 1 square foot (sq. ft.).
9 square feet	= 1 square yard (sq. yd.).
$30\frac{1}{4}$ square yards	= 1 square rod (sq. rd.).
160 square rods	= 1 acre (A.).
640 acres	= 1 square mile (sq. mi.).
1 mile square	= 1 section.
36 square miles	= 1 township.

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100 square feet = 1 square (of roofs, etc.).

Cubic Measure.

1728 cubic inches (cu. in.) = 1 cubic foot (cu. ft.).
27 cubic feet = 1 cubic yard (cu. yd.).

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128 cubic feet = 1 cord. The word "cord" is generally used, however, to mean a pile of wood 8 ft. long and 4 ft. high, the price depending (other things being the same) on the length of the stick.

1 cubic yard = 1 load (of earth, etc.).
$24\frac{3}{4}$ cubic feet = 1 perch.

English Money.

12 pence (d.) = 1 shilling (s.) = \$0.243 +.
20 shillings = 1 pound (£) = \$4.8665.

French Money.

100 centimes = 1 franc (fr.) = \$0.193.

The French system is also used in several other countries, as in Belgium, Switzerland, Italy, etc., but the names are not uniform, in Italy, for example, the franc being called a lira.

German Money.

100 pfennigs = 1 mark (M.) = \$0.238.

Apothecaries' Weight.

20 grains (gr.) = 1 scruple (sc. or ℈).
3 scruples = 1 dram (dr. or ʒ).
8 drams = 1 ounce (oz. or ℥).
12 ounces = 1 pound (lb.).
5760 grains = 1 pound.

The table of apothecaries' weight is used in selling drugs at retail.

Troy Weight.

24 grains (gr.) = 1 pennyweight (pwt. or dwt.).
20 pennyweights = 1 Troy ounce.
12 Troy ounces = 1 Troy pound.
437.5 grains = 1 Avoirdupois oz.
7000 grains = 1 Avoirdupois lb.
480 grains = 1 Troy oz.
5760 grains = 1 Troy lb.

Troy weight is used for precious metals.

I. Compound Numbers.

When a concrete number is expressed in several denominations it is called a *compound number.*

E.g., 3 ft. 6 in. But 3.5 ft. and $2.25 are not compound numbers.

Reduction of compound numbers is a process so familiar to the student that two examples will satisfy for illustration.

(1) *Reduction descending.* Reduce 365 da. 5 hrs. 48 mins. to minutes.

Explanation and solution. 1. $\because$ 1 da. = 24 hrs.
2. $\therefore$ 365 da. = 365 × 24 hrs. = 8760 hrs.
3. 8760 hrs. + 5 hrs. = 8765 hrs.
4. $\because$ 1 hr. = 60 mins.
5. $\therefore$ 8765 hrs. = 8765 × 60 min. = 525,900 mins.
6. 525,900 mins. + 48 mins. = 525,948 mins.

Practical calculation. (The notes in parentheses explain the operations.)

	365	
(Multiply by 3)=	1095	(For 24 = 3 × 8.)
(Multiply by 8 and add 5)=	8765	(*i.e.*, 365 × 24 hrs. + 5 hrs.)
(Multiply by 60 and add 48)=	525,948	(*i.e.*, 8765 × 60 mins. + 48 mins.)

(2) *Reduction ascending.* Reduce 525,948 mins. to days, etc.

Explanation and solution. 1. $\because$ 1 min. = $\frac{1}{60}$ hr.
2. $\therefore$ 525,948 mins. = 525,948 × $\frac{1}{60}$ hr. = 8765 hrs. and $\frac{48}{60}$ hr. or *48 min.*
3. $\because$ 1 hr. = $\frac{1}{24}$ da.
4. $\therefore$ 8765 hrs. = 8765 × $\frac{1}{24}$ da. = *365 da.* and $\frac{5}{24}$ da., or *5 hr.*
5. $\therefore$ 365 da. 5 hrs. 48 mins.

Practical calculation. (The notes in parentheses explain the operations.)

60	525948		
24	8765	48	(525,948 × $\frac{1}{60}$ hr.)
	365	5	(8765 × $\frac{1}{24}$ da.)

Exercises. 1. Reduce 43 wks. 5 hrs. 49 mins. 57 secs. to seconds.

2. Reduce 4,568,657 secs. to weeks, days, etc.
3. Reduce 625 cu. yds. 19 cu. ft. 1609 cu. in. to cubic inches.
4. Reduce 1,847,638 ft. to miles, yards, and feet.
5. Reduce 25 mi. 459 yds. 31 in. to inches.
6. Reduce 12,563,257 sq. in. to acres, square yards, etc.
7. Reduce 5 gals. 3 pts. to pints.
8. Reduce 341 qts. to gallons.
9. Reduce 150 lbs. to ounces. (Avoirdupois.)
10. Reduce 274 oz. to pounds and ounces; to pounds and decimals of a pound. (Avoirdupois.)
11. Reduce 36 gt. gro. to gross; to units.
12. Reduce 15 reams to quires; to sheets.
13. Reduce 142,872 sheets to quires; to reams.
14. Reduce 19,436 cu. ft. to cubic yards.
15. Reduce 2 wks. 2 da. 19.2 hrs. to the fraction of a month of 4 wks.
16. Reduce 14 hrs. 15 mins. to the fraction of $3\frac{1}{2}$ da.
17. Reduce $\frac{8}{11}$ pt. to the fraction of a gallon.
18. Reduce 3 qts. 1 pt. to the decimal of a gallon.
19. Reduce 18 hrs. 30 mins. 30 secs. to the fraction of a week.

Compound addition and subtraction differ so little in theory from the addition and subtraction of simple abstract numbers as to require no extended review. The cases arising in actual practice rarely involve more than two denominations, the tendency being to reduce the lower units to decimals of the higher. Thus, while it was formerly not unusual to add numbers like 27 rds. 5 yds. 2 ft. 11 in., it is now more common to deal with numbers like 463.4 ft., 1.27 A., 4.345 mi., etc.

9 lbs.	15 oz.
10	12
8	6
29	1

29 lbs.	1 oz.
10	12
18	5

In the annexed example in addition the computer should say, "6, 18, 33, **1**; 2, 10, 20, **29**." In the example in subtraction he should proceed as with simple numbers, remembering that 16 oz. = 1 lb., and should say, "12 and **5** are 17; 11 and **18** are 29."

Exercises. 1. What check should be used in compound addition? in compound subtraction?

2. Add 9 lbs. 7 oz., 52 lbs. 6 oz., 91 lbs. 12 oz., 7 lbs., 5 lbs. 2 oz., 13 oz. (Avoirdupois.)

3. Add 13 t. 450 lbs., 12 t. 700 lbs., 342 t., 44 t. 1500 lbs., 1200 lbs.

4. Add 25 gals. 3 qts., 47 gals. 2 qts. 1 pt., 15 gals. 1 qt., 1 pt., 9 gals.

5. Add 5 yds. 2 ft., 6 yds. 1 ft. 7 in., 9 yds. 2 ft. 5 in., 1 ft.

6. Add 6 bu. 3 pks., 9 bu. 2 pks., 5 bu. 1 pk., 3 pks.

7. Add 10 da. 5 hrs. 42 mins. 7 secs., 23 hrs. 10 mins. 2 secs., 11 da. 4 mins., 5 hrs. 4 mins. 5 secs., 1 da. 15 secs.

8. Add 7 mo. 15 da., 5 mo. 14 hrs. 3 secs., 1 mo. 7 da. 5 mins. 57 secs., 2 mo. 54 mins., 9 hrs., 7 da., 2 mo.

9. Add 12 mi. 3 rds. 2 ft., 3 mi. 75 rds. 10 ft., 4 mi. 12 ft., 3 rds. 6 ft.

10. From 13 lbs. 9 oz. subtract 9 lbs. 10 oz. (Avoirdupois.)

11. From 7 mo. 9 da. subtract 5 mo. 15 da.

12. From 25 gals. 2 qts. subtract 10 gals. 3 qts. 1 pt.

13. From 5 yds. 1 ft. 7 in. subtract 2 yds. 2 ft. 10 in.

14. From 6 bu. 2 pks. subtract 5 bu. 3 pks.

15. From 7 da. 5 hrs. 27 mins. 42 secs. subtract 5 da. 5 hrs. 27 mins. 43 secs.

16. From 87 cu. yds. 8 cu. ft. 924 cu. in. subtract 35 cu. yds. 23 cu. ft. 1688 cu. in.

Compound multiplication. The remarks already made concerning practical problems in addition and subtraction apply with equal force to multiplication and division. The general theory is evident from the analysis of the following problem.

Required the product of 10 × 3 lbs. 4 oz.

Analysis. 1. 10 × 4 oz. = 40 oz. = 2 lbs. 8 oz.

2. 10 × 3 lbs. = 30 lbs.

3. 30 lbs. + 2 lbs. 8 oz. = 32 lbs. 8 oz.

CALCULATION.

3 lbs. 4 oz.
10
32 lbs. 8 oz.

Compound division. The definition of division, already given on p. 29, should now be recalled for the purpose of distinguishing between the two general cases.

Division is the operation of finding one of two factors, the quotient, when the product and the other factor are given.

Hence, there are two general cases illustrated by the following example.

Since \$10 is the product of the factors 2 and \$5,

1. $\therefore$ \$10 ÷ \$5 = 2, the idea of *measuring*, *being contained in*, *continued subtraction.* That is, \$5 is contained in \$10 2 times.

2. \$10 ÷ 2 = \$5, the idea of *separation*, *partition*, *multiplication by a fraction.* That is, \$10 divided by 2 equals \$5, \$10 has been separated into two parts.

I. *When dividend and divisor are compound numbers of the same kind, they may be reduced to the same denomination and the division performed in the ordinary way.*

For example, how many times does 32 lbs. 8 oz. contain 3 lbs. 4 oz.? In this case it is more simple to reduce to pounds, thus:

1. 32 lbs. 8 oz. = 32.5 lbs.
2. 3 lbs. 4 oz. = 3.25 lbs.
3. 32.5 lbs. ÷ 3.25 lbs. = 10.

But it is usually easier to reduce to one of the lower denominations, especially where more than two denominations are involved. For example, how many times does 29 t. 87 lbs. 2 oz. contain 3 t. 4 cwt. 54 lbs. 2 oz.?

1. 29 t. 87 lbs. 2 oz. = 58087.125 lbs.
2. 3 t. 4 cwt. 54 lbs. 2 oz. = 6454.125 lbs.
3. 58087.125 lbs. ÷ 6454.125 lbs. = 9.

II. *When the divisor is an abstract number.*

For example, divide 29 t. 87 lbs. 2 oz. by 9.

CALCULATION.

$$9\,\underline{)\,29\text{ t. } 87\text{ lbs. } 2\text{ oz.}}$$
$$3\text{ t. } 454\text{ lbs. } 2\text{ oz.}$$

Analysis. 1. 29 t. ÷ 9 = *3 t.*, and 2 t., or 4000 lbs., remainder.

2. 4000 lbs. + 87 lbs. = 4087 lbs.

3. 4087 lbs. ÷ 9 = *454 lbs.*, and 1 lb., or 16 oz., remainder.

4. 16 oz. + 2 oz. = 18 oz.

5. 18 oz. ÷ 9 = *2 oz.*

6. $\therefore$ 3 t. 454 lbs. 2 oz.

Exercises. 1. Multiply 27 gals. 3 qts. 1 pt. 3 gi. by 36, checking by division.

2. Also by 236, checking by division.

3. Multiply 17 wks. 4 da. 13 hrs. 27 mins. 36 secs. by 9, checking by division.

4. Also by 79, checking by division.

5. Multiply 23 cu. yds. 6 cu. ft. 459 cu. in. by 8, checking by division.

6. Multiply the result in Ex. 5 by 9, checking by division.

7. Multiply 512 rds. 2 yds. 2 ft. 2 in. by 6, checking by division.

8. Multiply 2 sq. yds. 3 sq. ft. 9 sq. in. by 10, checking by division.

9. Divide 878 wks. 4 da. 15 hrs. 37 mins. 36 secs. by 56, checking by multiplication.

10. Divide 4285 cu. yds. 6 cu. ft. 1689 cu. in. by 23, checking by multiplication.

11. Also by 85, checking by multiplication.

12. Divide 5863 gals. 3 qts. 1 pt. 3 gi. by 8, checking by multiplication.

13. Also by 75, checking by multiplication.

14. How many jars, each containing 2 gals. 3 qts. 1 pt. 3 gi., can be filled from a cask containing 285 gals.? Check the result by multiplication.

15. Divide 346 da. 18 hrs. 34 mins. 32 secs. by 1 da. 7 hrs. 45 mins. 56 secs., checking the result by multiplication.

16. A carriage wheel revolves 3 times in going 11 yds.; how many times will it revolve in going $\frac{3}{4}$ of a mi.?

17. If 277,280 cu. in. of water weigh 10,000 lbs., how many cubic feet (approximately) will weigh 1000 oz.?

18. If a clock gains 12 mins. a day, what is the average gain per min.?

19. Supposing the distance traveled by the earth about the sun to be 596,440,000 mi., what is the average hourly distance traveled, taking the year to equal $365\frac{1}{4}$ da.?

20. Supposing the distance from the earth to the sun to be 91,713,000 mi. and that the sun's light reaches the earth in 8 mins. 18 secs., what is the velocity of light per sec.?

21. From the data of Ex. 20 and that on p. 3, find how long it would take the sun's light to reach Neptune. Express the result in hours, minutes, and seconds.

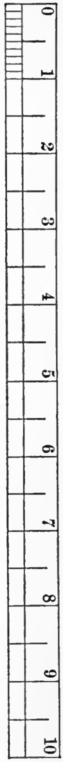
II. The Metric System.

Soon after the opening of the nineteenth century, France legalized a uniform system of measures generally known as the Metric System. This is now used in practical business by most of the highly civilized nations, except the United States and England and her dependencies. In scientific work it is generally used by all countries, and there is every reason to believe that it will also become universal in business.

Units. The system is based on the unit of length, called the *meter* (meaning *measure*), which is 0.0000001 (or 10^{-7}) of the distance from the equator to the pole.

The unit of capacity is the *liter*, a cube 0.1 of a meter on an edge.

The unit of weight is the *gram*, the weight of a cube of water 0.01 of a meter on an edge.



The annexed scale shows the decimeter divided into centimeters, one centimeter being divided into millimeters.

Through an error in fixing the original units, they are not exactly as stated, but the system loses none of its practical advantages on this account. The original units are preserved at Paris.

The prefixes set forth on p. 60 must be thoroughly memorized, after which the metric system offers few difficulties. Some of these prefixes are never used with certain units in practice, just as the only units generally used in speaking of United States money are dollars and cents. We never say, "4 eagles 2 dollars 5 dimes and 3 cents" for $42.53. So in the metric system the myrialiter and kiloliter are never used, and the dekameter and hektometer rarely. In the following tables, the units most commonly used are, therefore, printed in bold-faced type.

The abbreviations of the metric system are not uniform even in France. Those here given have been adopted by the International Committee of Weights and Measures and by other international associations, and are therefore given as the most approved now in use.

TABLES.

	THE PREFIX	MEANS	AS IN	WHICH MEANS	
From the Greek.	myria-	10000	myriameter	10000	meters.
	kilo-	1000	kilogram	1000	grams.
	hekto-	100	hektoliter	100	liters.
	deka-	10	dekameter	10	meters.
		1		1	
From the Latin.	deci-	0.1	decimeter	0.1	of a meter.
	centi-	0.01	centigram	0.01	of a gram.
	milli-	0.001	millimeter	0.001	of a meter.
Greek.	mikro-	0.000001	mikrometer	0.000001	of a meter.

TABLE OF LENGTH.

A myriameter	=	10,000	meters.
A **kilometer** (km)	=	1000	"
A hektometer	=	100	"
A dekameter	=	10	"
Meter (m)			
A decimeter (dm)	=	0.1	of a meter.
A centimeter (cm)	=	0.01	"
A **millimeter** (mm)	=	0.001	"
A **mikron** (μ)	=	0.000001	"

TABLE OF SQUARE MEASURE.

A square myriameter	=	100,000,000	square meters.
" kilometer (km^2)	=	1,000,000	"
" hektometer	=	10,000	"
" dekameter	=	100	"
Square meter (m^2)			
A square decimeter (dm^2)	=	0.01	of a square meter.
" centimeter (cm^2)	=	0.0001	"
" millimeter (mm^2)	=	0.000001	"

The square dekameter is also called an **are**; and since there are 100 dm^2 in 1 hm^2, a square hektometer is called a **hektare**. These are used in measuring land.

TABLE OF CUBIC MEASURE.

A cubic myriameter	=	10^{12}	cubic meters.
" kilometer	=	10^9	"
" hektometer	=	1,000,000	"
" dekameter	=	1000	"
Cubic meter (m^3)			

A cubic decimeter (dm^3)	=	0.001	of a cubic meter.
" centimeter (cm^3)	=	0.000001	"
" millimeter (mm^3)	=	0.000000001	"

The cubic meter is also called a **stere**, a unit used in measuring wood.

TABLE OF WEIGHT.

A **metric ton** (t)	=	1,000,000	grams.
A quintal (q)	=	100,000	"
A myriagram	=	10,000	"
A **kilogram** (kg)	=	1000	"
A hektogram	=	100	"
A dekagram	=	10	"
Gram (g)			
A decigram	=	0.1	of a gram.
A centigram (cg)	=	0.01	"
A milligram (mg)	=	0.001	"
A mikrogram (γ)	=	0.000001	"

The metric ton is the weight of 1 m^3 of water; the kilogram of 1 dm^3 or 1 liter of water; and the gram of 1 cm^3 of water.

TABLE OF CAPACITY.

A hektoliter (hl)	=	100	liters.
A dekaliter	=	10	"
Liter (l)			
A deciliter	=	0.1	of a liter.
A centiliter	=	0.01	"
A milliliter (ml)	=	0.001	"
A mikroliter (λ)	=	0.000001	"

TABLE OF EQUIVALENTS.

In general, the metric system is used by itself, as in scientific work, and the common English-American system by itself. Hence, there is little demand for reducing from one to the other. Such reductions are, however, occasionally necessary, and hence a few of the common equivalents are here given. These equivalents are only approximate.

A meter	= 39.37 inches	= $3\frac{1}{4}$ feet nearly.
A liter	= 1 quart nearly.	
A kilogram	= 2.2 pounds nearly.	
A kilometer	= 0.62 of a mile	= 0.6 of a mile nearly.
A gram	= 15.43 grains	= $15\frac{1}{2}$ grains nearly.
A hectare	= 2.47 acres	= $2\frac{1}{2}$ acres nearly.

Oral Exercises. 1. What is the meaning of hekto-? myria-? centi-? kilo-? deci-? deka-? milli-? mikro-?

2. What is the prefix which means 10,000? 0.1? 10? 100? 0.01? 1000? 0.001?

3. About when was the metric system established? Where? How extensively is it used at present, (*a*) in business, (*b*) in science? What are its advantages over the older systems?

4. How was the length of the meter fixed? How was the liter fixed? the gram?

5. What is the weight of a liter of water? (This is only approximate and it refers to distilled water at its maximum density.)

6. What is the weight of a cubic centimeter of water? of a cubic decimeter? of a cubic meter?

7. How many mm in a km? in a hektometer? in a myriameter?

8. How many cm^2 in a m^2? in a km^2?

9. How many mm^3 in a cm^3? in a liter? in a m^3?

10. How many g in 125 kg? in a metric ton?

11. How many dm^3 in 5 steres? in a cubic dekameter?

12. Reduce 17 km to m; to mm; to dm; to μ.

13. Reduce 5 dekaliters to l; to cm^3; to λ.

14. Reduce 300 ha to a; to m^2; to km^2.

15. Reduce 45,000 m^2 to ha; to a; to a fraction of a km^2.

16. Reduce 0.573 m^2 to cm^2; to mm^2.

17. Reduce 15 km^2 to m^2; to cm^2; to ha.

18. Reduce 27 m^3 to dm^3; to mm^3; to l.

19. 25 kg are how many lbs., to the nearest unit?

20. 300 km are how many mi., to the nearest unit?

21. 65 l are how many qts., to the nearest unit?

22. 30 ha are how many acres, to the nearest unit?

23. 50 acres are how many ha, to the nearest unit?

24. 20 qts. are how many l, to the nearest unit?

25. 50 mi. are how many km, to the nearest unit?

26. 220 lbs. are how many kg, to the nearest unit?

27. 325 ft. are how many m, to the nearest unit?

28. The Eiffel tower at Paris is 300 m high; this is about how many feet?

29. The papers report the rainfall at Berlin, for a given period, to be 11.0 cm; this is how many inches, to the nearest tenth?

30. What is the pressure in grams per cm^2 of a column of water 1 m deep?

Written Exercises. 1. The length of a wave of sodium light is 5893×10^{-8} cm; how many such wave-lengths in 1 m? in 1 μ?

2. Cast copper being 8.8 times as heavy as an equal volume of water, what is the weight of 5 dm^3?

3. A stream flowing uniformly 1 km per hr. flows how many cm per sec.?

4. A liter of mercury weighs 13.596 kg; how many mm^3 of mercury weigh 1 g?

5. A man takes 120 steps in walking 100 m; what is the average length of each step?

6. At the rate given in Ex. 5, how many steps will be taken in walking 6.2 km?

7. How many mm^2 in $\frac{3}{4}$ dm^2? in $\frac{3}{4}$ of a decimeter square?

8. Air being 0.001276 as heavy as an equal volume of water, what is the weight of air in a room containing 600 m^3?

9. A train traveling 1 km per min. travels how many m per sec.?

10. Sound travels 332 m per sec.; how long will it take it to travel 1 km? Answer to 0.01 sec.

11. Granite being 2.7 times as heavy as water, what is the weight of a block containing 2.50 m^3?

12. Show that an inch is nearly 2.54 cm, and use this equivalent in Exs. 13, 14.

13. Express the following readings from a barometer in centimeters: 29.9 in., 30.0 in., 30.1 in., 30.2 in.

14. Also the following in inches, to the nearest 0.1: 71.119 cm, 73.659 cm, 74.929 cm.

15. Olive oil being 0.92 as heavy as an equal volume of water, and petroleum 0.7, and alcohol 0.83, what is the weight of a liter of each?

16. The distance from Paris to Rouen is 136 km; the prices of tickets are, 1st class 15.25 francs, 2d class 11.40 francs, 3d class 8.40 francs; what is the price for each class per kilometer?

17. The United States government lays down these regulations concerning foreign mail: letters weighing 15 g or less require 5 cts.; packets of samples of merchandise may be sent not exceeding 350 g in weight, 30 cm in length, 20 cm in width, and 10 cm in height; express these measures in the common units.

18. The Eiffel tower at Paris is 300 m high and cost about 5,000,000 francs; if enough 20-franc gold pieces each $1\frac{1}{4}$ mm thick could be piled one above another to equal this sum, would the pile equal the height of the tower?

III. Measures of Temperature.

Temperature is ordinarily measured by a thermometer using one of two scales, (*a*) the Fahrenheit, used in this country in ordinary business, or (*b*) the Centigrade, quite generally used in science.

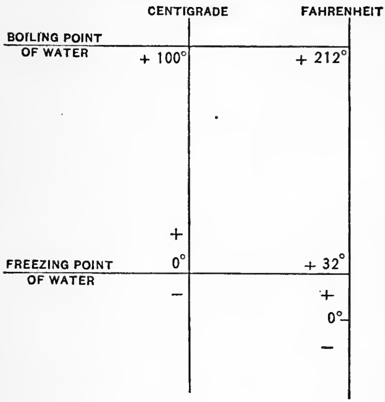


Fig. 1.

The Fahrenheit (Fah.) scale was suggested by Fahrenheit in the early part of the eighteenth century. He took for 0° a temperature which he obtained by mixing ice and salt, and for 212° the boiling point of water, thus bringing the freezing point at 32°, a very unscientific arrangement.

The Centigrade (C., from Latin *centum*, hundred, and *gradus*, degree) scale was adopted by Celsius in 1742, and is often called by his name. It places 0° at the freezing, and 100° at the boiling point of water.

Degrees above 0° on each scale are usually indicated by the sign +; below, by the sign −.

To reduce from Fahrenheit to Centigrade.

$\because$ 212° Fah. − 32° Fah., or 180° Fah. = 100° C.,

$\therefore$ 1° Fah. $= \dfrac{100° \text{ C.}}{180}$, or $\dfrac{5° \text{ C.}}{9}$. It must also be remembered that the freezing point of water is at + 32° Fah.

E.g., + 80° Fah. corresponds to what temperature C.?

1. + 80° Fah. = + 80° Fah. − 32° Fah. above freezing, or 48° Fah. above freezing.

2. 1° Fah. $= \dfrac{5° \text{ C.}}{9}$,

3. $\therefore$ 48° Fah. $= 48 \times \dfrac{5° \text{ C.}}{9} = 26.67°$ C.

Also, $-10°$ Fah. corresponds to what temperature C.?

1. $-10°$ Fah. $-32°$ Fah. $= -42°$ Fah.
2. $1°$ Fah. $= \frac{5° \text{ C.}}{9}$,
3. $\therefore -42°$ Fah. $= -42 \times \frac{5° \text{ C.}}{9} = -23.33°$ C.

To reduce from Centigrade to Fahrenheit.

$\because 100°$ C. $= 180°$ Fah.

$\therefore \quad 1°$ C. $= 1.8°$ Fah.

E.g., $+80°$ C. corresponds to what temperature Fah.?

1. $1°$ C. $= 1.8°$ Fah.
2. $\therefore 80°$ C. $= 80 \times 1.8°$ Fah. $= 144°$ Fah. above freezing.
3. $\therefore 80°$ C. corresponds to $144°$ Fah. $+ 32°$ Fah., or $176°$ Fah.

Exercises. Find the temperatures, Fah. or C., corresponding respectively to the following temperatures, C. or Fah.

1. $+122°$ Fah.	2. $+115.8°$ Fah.	3. $-35°$ Fah.
4. $-30°$ Fah.	5. $+30°$ Fah.	6. $+25.7°$ Fah.
7. $+13.1°$ Fah.	8. $-40.7°$ Fah.	9. $+100.5°$ Fah.
10. $-1.2°$ Fah.	11. $+1.2°$ Fah.	12. $+100°$ Fah.
13. $+40°$ C.	14. $-40°$ C.	15. $+100°$ C.
16. $-7.8°$ C.	17. $+15.8°$ C.	18. $-17.78°$ C.
19. $-21.4°$ C.	20. $-18.1°$ C.	21. $-45°$ C.
22. $+45°$ C.	23. $-33°$ C.	24. $-9.6°$ C.

25. Express on Centigrade scale the following melting points: (*a*) lead $+630°$ Fah., (*b*) mercury $-38.99°$ Fah., (*c*) ice $+32°$ Fah., (*d*) silver $+873°$ Fah., (*e*) copper $+1996°$ Fah., (*f*) cast iron $+2786°$ Fah., (*g*) tin $+455°$ Fah.

26. Express on Fahrenheit scale the following boiling points: (*a*) alcohol $+78°$ C., (*b*) ether $+35°$ C., (*c*) mercury $+357°$ C., (*d*) sulphuric acid $+338°$ C.

27. There is another kind of thermometer which is often used, known as the Réaumur thermometer, the $0°$ being the freezing and $80°$ being the boiling point of water. Express on the Réaumur scale the following melting points: (*a*) camphor $175°$ C., (*b*) paraffine $55°$ C., (*c*) phosphorus $44°$ C., (*d*) rock salt $800°$ C., (*e*) sugar crystal $170°$ C.

28. Express on the Réaumur scale the following boiling points: (*a*) wood alcohol $150.8°$ Fah., (*b*) benzine $176°$ Fah., (*c*) chloroform $141.8°$ Fah., (*d*) glycerine $554°$ Fah.

CHAPTER VII.

Mensuration.

THERE are certain measurements which are so commonly needed in business and in science that they are generally considered as part of arithmetic, although the strict proofs of the principles involved are matters of geometry. It is, however, possible to arrive at the results without the use of demonstrative geometry, and for those who have not studied that science the present chapter will be of value.

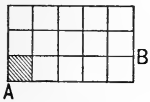


FIG. 2.

Area of a rectangle. If two sides of a rectangle are 3 in. and 5 in. respectively, then, in the figure, the area of the strip AB is 5×1 sq. in., and the total area is $3 \times 5 \times 1$ sq. in., or 15 sq. in.

Similarly, the area of a rectangle b cm long and h cm high is $h \cdot b \cdot 1$ cm², or hb cm². This is often expressed by saying that the area equals the product of the base and altitude, meaning that the abstract numbers have this relation.

Length of a rectangle. If a rectangle has an area of 6 cm² and a height of 2 cm, the *number* of units of length, say l, must be such that $l \cdot 2 \cdot 1 \text{ cm}^2 = 6 \text{ cm}^2$;

$$\therefore l = \frac{6 \text{ cm}^2}{2 \cdot 1 \text{ cm}^2} = 3\,; \therefore \text{ the length is 3 cm.}$$

Similarly, if the area is a cm² and the height is h cm, the *number* of units of length, say l, must be such that $l \cdot h \cdot 1 \text{ cm}^2 = a \text{ cm}^2$;

$$\therefore l = \frac{a \text{ cm}^2}{h \cdot 1 \text{ cm}^2} = \text{an abstract number expressing the units of length.}$$

Area of any parallelogram. Since any parallelogram has the same area as the rectangle of the same base and altitude, as is seen by cutting off the triangle T in the figure and placing it in the position T′, for purposes of measurement of area, length, and height, a parallelogram may be considered a rectangle.

FIG. 3.

Area of a triangle. Since two congruent triangles, as T in the figure, may be so placed as to form a parallelogram, therefore any triangle is half of a rectangle of the same base and altitude.

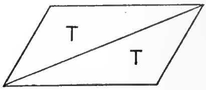


FIG. 4.

Hence, the area of a triangle of base 3 cm and altitude 2 cm is $\frac{1}{2} \cdot 2 \cdot 3 \cdot 1 \text{ cm}^2 = 3 \text{ cm}^2$. In general, if the base is b cm and the height is h cm, the area is $\frac{1}{2}hb \text{ cm}^2$.

If the area of a triangle is 3 cm² and the height is 2 cm, the *number* of units of length of base, say b, is such that $\frac{1}{2} \cdot 2 \cdot b \cdot 1 \text{ cm}^2 = 3 \text{ cm}^2$;

$\therefore b = \dfrac{3 \text{ cm}^2}{\frac{1}{2} \cdot 2 \cdot 1 \text{ cm}^2} = 3$; therefore the base is 3 cm.

Area of a trapezoid. If the trapezoid T in the figure swings about the point o to the position T′, leaving its trace at T, the figure TT′ is a parallelogram. Hence, the area of a trapezoid equals half the area of a rectangle of the same altitude and with a base equal to the sum of the two bases of the trapezoid.

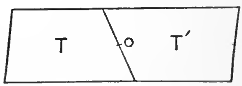


FIG. 5.

Exercises. 1. Find the area of a parallelogram of which the base is 4 ft. 3 in. and the altitude 4 ft.

2. Find the area of the right-angled triangle in which the two sides including the right angle are 5 ft. and 8 ft.

3. Find the area of a trapezoid whose bases are 6 ft. 2 in. and 7 ft. 4 in. and whose altitude is 4 ft.

Abscissas and ordinates. If, in Fig. 6, XX' is perpendicular to YY' at O, and y_1 to OX, then x_1 is called the *abscissa* of point P_1, and y_1 is called the *ordinate* of that point.

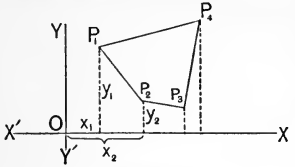


FIG. 6.

Similarly x_2, y_2, are the abscissa and ordinate of P_2, and so on for P_3, P_4, etc.

To find the area of the field $P_1P_2P_3P_4$, the area of the trapezoid between y_1 and y_4 may be found, and from this may be subtracted the areas of the trapezoids between y_1 and y_2, y_2 and y_3, y_3 and y_4; this plan is sometimes used by surveyors, OY representing the north line and OX the east line. It is more convenient, however, to let OX pass through the most southern point P_3, or OY pass through the most western point P_1. Surveyors usually take the latter plan and find the areas of the trapezoids between x_3 and x_4, x_3 and x_2, etc.

Mensuration of a square. Since a square is a particular kind of rectangle, the area of a square of side 3 cm is $3 \cdot 3 \cdot 1 \text{ cm}^2 = 9 \text{ cm}^2$. And if the area is given as 9 cm^2, the *number* of units in a side, say s, is such that $s \cdot s \cdot 1 \text{ cm}^2 = 9 \text{ cm}^2$, whence $s^2 = 9$, and $s = \sqrt{9} = 3$. $\therefore$ the side is 3 cm long.

It should again be observed that, by the ordinary definition of square root, the expression $\sqrt{9 \text{ cm}^2}$ has no meaning.

The Pythagorean theorem. In the figure, if the triangles 1, 2, 3, 4 are taken away, the square on the hypotenuse of a right-angled triangle remains; and if the two rectangles AP, PB, are taken away from the whole figure, the sum of the squares on the two sides of the triangle remains; but the four triangles together equal the two rectangles, hence *in a right-angled triangle the sum of the squares on the two sides equals the square on the hypotenuse.*

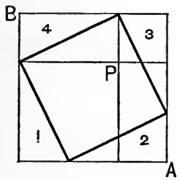


FIG. 7.

This is known in geometry as the Pythagorean theorem because it is supposed to have been first proved by Pythagoras (about 500 B.C.).

If the sides of a right-angled triangle are 3 ft. and 4 ft., then

1. $3 \cdot 3 \cdot 1$ sq. ft. = 9 sq. ft., the square on one side, and
2. $4 \cdot 4 \cdot 1$ sq. ft. = 16 sq. ft., the square on the other side.
3. $\therefore$ 9 sq. ft. + 16 sq. ft. = 25 sq. ft., the square on the hypotenuse.
4. $\therefore$ the *number* of units in the hypotenuse $= \sqrt{25} = 5$.
5. $\therefore$ the hypotenuse is 5 ft. long.

The word "equal" as here used means "have the same area as." For full discussion of the words "equal" and "equivalent," see Beman and Smith's "Plane and Solid Geometry," p. 20.

Exercises. 1. What is the area of a floor 26 ft. 4 in. long and 42 ft. 8 in. wide?

2. What is the length of a rectangular field 370 ft. wide, containing an area of 7850 sq. rds.?

3. What is the length, to 0.01 m, of a square whose area is 50 m^2?

4. The area of a triangle is 57 sq. in. and the height is 5 in.; find the length of the base.

5. Find, to 0.001 in., the altitude of an equilateral (equal sided) triangle whose side is 4 in.; also, to 0.01 sq. in., the area.

6. Find the length of one of the equal sides of an isosceles triangle whose base is 8 cm and altitude 3 cm. (An isosceles triangle is a triangle having two equal sides.)

7. The area of an equilateral triangle is 10 sq. in.; find, to 0.001 in., the length of a side.

8. A ship is sailing at the rate of 11.25 mi. per hr.; a sailor climbs a mast 55 yds. high in 24 secs.; find his rate of motion.

9. A ship is sailing at the rate of 12 mi. per hr.; a sailor walks across the deck at the rate of 5 mi. per hr.; find his rate of motion.

10. A man swims at right angles to the bank of a stream at the rate of 3.6 mi. per hr.; the rate of the current is 10.5 mi. per hr.; find the rate of the swimmer's motion.

11. In Ex. 10, suppose the stream to be 972 ft. wide; how far down stream is the swimmer carried by the current?

12. A square and an equilateral triangle have the same perimeter; how do their areas compare? (Assume the side of the square to be s, find the side of the triangle, and then find the area of each.)

13. The hypotenuse of a right-angled triangle is 5 ft., and one side is 4 ft.; show that the equilateral triangle on the hypotenuse equals the sum of the equilateral triangles on the two sides.

14. How many square yards of plain paper are necessary to cover the walls and ceiling of a room 20 ft. by 15 ft. and 10 ft. high, allowance being made for three windows, each 5 ft. 4 in. by 3 ft., and one door 8 ft. by 3 ft.?

15. The abscissas of the several corners of a field are (in chains) 1.00, 2.25, 6.00, 7.50, 4.30, and the corresponding ordinates are 5.10, 0.90, 2.00, 4.70, and 7.00. Draw the figure to a scale and compute the area in acres.

16. A gardener has a yard 150 ft. by 250 ft. to be laid out into beds; allowing a foot all around the edge of the yard for grass, how many beds 4 ft. by 8 ft. can be laid out?

17. How many bricks 4 in. by 8 in. are needed to lay a cellar floor 20 ft. by 36 ft.?

18. How many acres in a farm which consists of two rectangular pieces, one 95 rds. by 160 rds. and the other 75 rds. by 22 rds.?

19. A rectangular farm 205 rds. long and 175 rds. wide is under cultivation, with the exception of a piece of woods 62 rds. by 58 rds.; how many acres are cultivated?

20. How many bunches of shingles should be bought for shingling a barn with a 40-ft. roof and 16-ft. rafters, laying the shingles 4 in. to the weather and a double row at the bottom? The average width of the shingles is 4 in. and there are 250 in a bunch.

21. How many bushels of wheat would be taken from a field 60 rds. by 80 rds., the average yield being 30 bu. to the acre?

22. How much will it cost to plaster a room 25 ft. long, 20 ft. wide, and 12 ft. high, at 9 cts. per sq. yd., no allowance being made for windows, etc.?

23. It is shown in physics that if two forces are pulling from a point P and are represented in direction and intensity by the lines PA, PB, the resultant force is represented by PC, the diagonal of their parallelogram. Suppose two forces are pulling at right angles with the intensity 5 lbs. and 8 lbs., what is the intensity of the resulting force? Draw the parallelogram to some convenient scale.

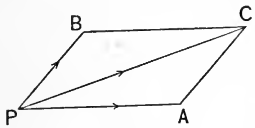


FIG. 8.

24. As in Ex. 23, suppose the forces are each 10 kg, pulling at an angle of 60° (an angle of an equilateral triangle).

25. As in Ex. 23, suppose the forces are each 3 lbs., pulling at an angle of 90°.

Board measure. In measuring lumber, a board 1 ft. long, 1 ft. wide, and 1 in. or less thick is said to have a measurement of 1 *board foot.*

E.g., a board 16 ft. long, 12 in. wide, and 1 in. or less thick contains 16 board feet. But a board 16 ft. long, 8 in. wide, and $1\frac{1}{2}$ in. thick contains $16 \cdot \frac{8}{12} \cdot \frac{3}{2}$ of 1 board foot, or 16 board feet.

In speaking of lumber, the word "foot" is generally used for "board foot." The price of lumber is usually quoted by the 1000 board feet; thus, "$25 per M" means $25 per 1000 board feet.

Exercises. 1. How many feet in each of the following boards:

(*a*) 14 ft. long, 8 in. wide, 2 in. thick?

(*b*) 16 ft. long, 6 in. wide, $1\frac{1}{2}$ in. thick?

(*c*) 12 ft. long, 4 in. wide, 2 in. thick?

(*d*) 15 ft. long, 9 in. wide, $\frac{3}{4}$ in. thick?

(*e*) 10 ft. long, 12 in. wide, 1 in. thick?

(*f*) 8 ft. long, 10 in. wide, 2 in. thick?

(*g*) 12 ft. long, 8 in. wide, 1 in. thick?

(*h*) 16 ft. long, 6 in. wide, 1 in. thick?

2. What is the cost of 20 2-in. planks, each 16 ft. long by 9 in. wide, @ $25 per M?

3. What is the cost of 50 $1\frac{1}{2}$-in. boards, each 12 ft. long by 6 in. wide, @ $30 per M?

4. How many feet of lumber will it take to floor a barn 30 ft. long and 16 ft. wide with 2-in. planks?

5. How many feet in a beam 18 ft. long and 8 in. square on the end?

6. A 16-ft. beam costs $4 @ $30 per M; it is square on the end; what is the thickness?

7. How many feet in a stick of timber 16 ft. long and 10 in. square?

8. How much will it cost for 1-in. boards to fence an acre lot 8 rds. wide, the boards being 6 in. wide and the fence 4 boards high, the price of lumber being $20 per M?

9. How many feet of unmatched lumber will it take to cover the sides and gable peaks of a barn 30 ft. long and 20 ft. wide, 16 ft. high to the eaves, the gable peaks being 8 ft. high?

10. How many feet of lumber will it take to build a 6-board fence around a 3-acre rectangular lot 316.8 ft. wide, 6-in. boards being used and no allowance being made for waste in cutting?

Circumference of a circle. By measuring the circumferences and the diameters of several circles, and taking the average, the circumference will be found to be about $3\frac{1}{7}$ times the diameter. In geometry (see Beman and Smith's "Geometry," p. 190) it is proved that the circumference is more nearly 3.14159····· times the diameter. This number, 3.14159·····, is usually represented by the Greek letter π (*pī*).

For practical purposes, the value of π is usually taken as $3\frac{1}{7}$ or else as 3.1416. The latter value is used in this work and should be employed by the student in all computations unless otherwise directed.

$\because$ if c stands for the circumference, and d the diameter, and r the radius,

$$c = \pi d = 2\,\pi r.$$

$$\therefore d = c/\pi.$$

$\because c/\pi$ is the same as $\frac{1}{\pi}\cdot c$, it is often convenient to know $\frac{1}{\pi}$, the reciprocal of π. This is easily shown to be 0.3183·····, and may be used as a multiplier instead of using π as a divisor, if desired. Carried to seven decimal places the value is 0.3183098.

Area of a circle. A circle can be cut into figures which are nearly triangles with altitude r and with bases whose sum is c. Supposing the figures to be triangles, the area would be $\frac{1}{2}\cdot c\cdot r$, and it is proved in geometry that this is the exact area.

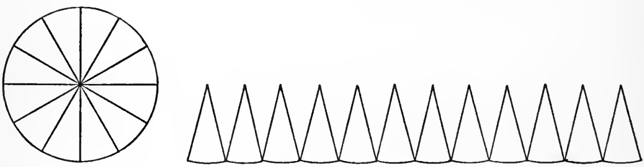


FIG. 8.

$\because c = 2\,\pi r$, $\therefore$ if a stands for the area,

$$a = \tfrac{1}{2}\,cr = \tfrac{1}{2}\cdot 2\,\pi r\cdot r = \pi r^2.$$

E.g., if the radius is 3 ft., the area is

$$\pi\cdot 3\cdot 3\ \ 1\text{ sq. ft.} = 28.2743\text{ sq. ft.}$$

Exercises. 1. Show that $r = c/2\pi$.

2. Show that $a = \frac{1}{4}\pi d^2$.

3. Show that $r = \sqrt{a/\pi}$, and hence that $d = 2\sqrt{a/\pi}$.

4. Find the circumferences of circles with radii (*a*) 22, (*b*) 24.1 in., (*c*) 17.1 m, (*d*) 0.123 m, (*e*) 34 mm, (*f*) 10.5 ft., (*g*) 4 yds., each correct to 0.001.

5. Find the radii of circles with circumferences (*a*) 311.0177 cm, (*b*) 207.3451 in., (*c*) 160.2212 m, (*d*) 43.9823 ft., (*e*) 1.2566 cm, each correct to 0.1.

6. Find the areas of circles with radii (*a*) 30 in., (*b*) 325 cm, (*c*) 17.8 mm, (*d*) 425 ft., (*e*) 0.78 m, each correct to 0.001.

7. Find the radii of circles with areas (*a*) 606,831 m², (*b*) 636,173 sq. ft., (*c*) 7.0686 cm², (*d*) 5026.5482 sq. in., (*e*) 1963.4954 sq. in., each correct to 0.1.

8. Find the areas of circles with circumferences (*a*) 163.3628 in., (*b*) 628.32 ft., (*c*) 889.07 cm, (*d*) 785.40 cm, (*e*) 2180.27 m, each correct to 1.

9. Find the circumferences of circles with areas (*a*) 372,845 sq. in., (*b*) 384,845 sq. ft., (*c*) 66,052 m², (*d*) 61,575 cm², (*e*) 7697.6874 sq. in., each correct to 0.1.

10. Find the diameter of a wheel that makes 373 revolutions in a mile. (Correct to 0.1.)

11. What should be the area of the opening of a cold-air box for a furnace to supply 1 hot-air pipe 1 ft. in diameter and 5 hot-air pipes 8 in. in diameter, it being necessary that the cross area of the cold-air box be $\frac{6}{7}$ that of the hot-air pipes together?

Volume of a rectangular parallelepiped. If a rectangular parallelepiped is 1 cm long, 1 cm wide, and 1 cm high, the volume is, by definition, 1 cm³; if it is 3 times as long, its volume is $3 \cdot 1$ cm³; and if it is also twice as high, its volume is $2 \cdot 3 \cdot 1$ cm³; and if it is also twice as wide, its volume is $2 \cdot 2 \cdot 3 \cdot 1$ cm³, or 12 cm³.

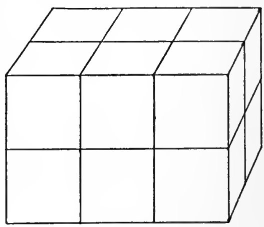


FIG. 9.

Similarly, the volume of such a solid l cm long, w cm wide, and h cm high is whl cm³.

Length of a rectangular parallelepiped. If the volume of the solid is 12 cm^3, and the area is 4 cm^2 on an end, the *number* of units of length must be such that $l \cdot 4 \cdot 1\ \text{cm}^3 = 12\ \text{cm}^3$; $\therefore\ l = \frac{12\ \text{cm}^3}{4 \cdot 1\ \text{cm}^3} = 3$; $\therefore$ the length is 3 cm.

Similarly, if the length, 3 cm, and the volume, 12 cm^3, are known, and the area of an end is required, the *number* of square units, a, must be such that $3 \cdot a \cdot 1\ \text{cm}^3 = 12\ \text{cm}^3$; $\therefore\ a = \frac{12\ \text{cm}^3}{3 \cdot 1\ \text{cm}^3} = 4$; $\therefore$ the area of an end is 4 cm^2.

Volume of any parallelepiped. As any parallelogram has the same area as the rectangle of equal base and equal altitude, so any parallelepiped has the same volume as the rectangular parallelepiped of equal base and equal altitude.

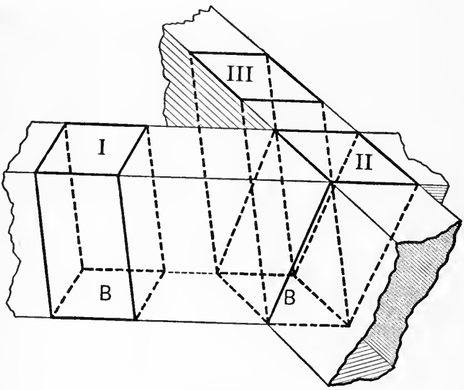


FIG. 10.

In the figure, solid III equals solid II, and solid II equals solid I, a rectangular parallelepiped of equal base and equal altitude.

The proof is too elaborate to be considered at this time. It is, however, somewhat similar to that already given as to the area of a parallelogram. See Beman and Smith's "Plane and Solid Geometry," p. 252.

Exercises. 1. What is the volume of a cube 1 in. on an edge? 2 in.? 4 in.? 8 in.?

2. A cube has a volume of 5 cu. ft.; find the edge, correct to 0.001 ft.

3. A parallelepiped of altitude 6 cm has a base 2 cm wide by 3 cm long; what is its volume? Similarly, if the dimensions are 4.75 cm, 0.35 cm, $3.33\frac{1}{3}$ cm.

4. On a base of 7.25 sq. ft. is a parallelepiped whose volume is 20 cu. ft.; find the height, correct to 0.001 ft. What would be the height if the volume were 40 cu. ft.?

5. What is the cost of digging a cellar 24 ft. by 12 ft., and 3 ft. deep, at 40 cts. per cu. yd.?

6. How many cubic feet of ice can I pack in an ice house 100 ft. by 63 ft., 18 ft. high, allowing 3 ft. on each side and 2 ft. above and below for sawdust?

7. At 66 cts. a bushel, what is the value of the wheat which fills a bin 6 ft. by 5 ft. by 5 ft.?

8. How many cords in a pile of 4-ft. wood 18 ft. long and 4 ft. high?

9. In a set of 12 steps to a high-school building each step is composed of 4 blocks of sandstone, each block being $2\frac{1}{2}$ ft. long, 2 ft. wide, $7\frac{1}{2}$ in. high; the stone costs 60 cts. per cu. ft., laid; find the cost of the steps.

10. How many loads (cubic yards) of earth must be taken out in excavating a canal 8200 ft. long, 300 ft. wide, and 16 ft. deep?

11. How many gallons (calling $7\frac{1}{2}$ gals. equal to 1 cu. ft.) in a tank 2 ft. by 2 ft. by 1 ft.?

12. A pile of 4-ft. wood 300 ft. long and 3 ft. 4 in. high cost $125; how much did it cost per cord?

13. Find the surface and the volume of a cube in which the diagonal of each face is 15 in.

14. The volume of a cube is 8000 cu. in.; required the length of its diagonal.

15. Find the edge of a cube whose surface equals the sum of the surfaces of two cubes whose edges are 120 in. and 209 in.

16. How many tons of water will a tank 16 ft. long, 8 ft. wide, and 7 ft. deep contain? (1 cu. ft. of water weighs 1000 oz. nearly.)

17. A zinc tank, open at the top, is 32 in. long, 21 in. wide, and 16.5 in. deep, inside measure; the metal is $\frac{1}{4}$ in. thick; required its weight and the weight of the water which it will hold. (Zinc is 7.2 times as heavy as water.)

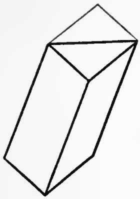


Fig. 11.

Volume of a prism. Since any triangular prism equals half of a parallelepiped of the same altitude and of double the base, therefore any triangular prism equals a parallelepiped of the same altitude and equal base, and hence it equals a rectangular parallelepiped of equal base and equal altitude.

And since any prism can be cut into triangular prisms, as in Fig. 12, therefore any prism equals a rectangular parallelepiped of equal base and equal altitude.

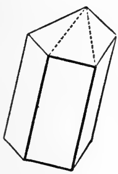


Fig. 12.

Volume of a pyramid. If a hollow prism and a hollow pyramid of the same base and altitude be made from pasteboard, and the latter be filled three times with sand and the contents poured into the prism, the latter will be exactly full. Hence it may be inferred, as it is proved in geometry, that the volume of a pyramid is one-third the volume of a rectangular parallelepiped of equal base and equal altitude.

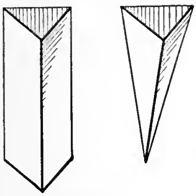


Fig. 13.

E.g., the volume of a prism of altitude 2 ft. and base 3 sq. ft. is $2 \cdot 3 \cdot 1$ cu. ft. $= 6$ cu. ft. The volume of a pyramid of equal base and equal altitude is $\frac{1}{3} \cdot 2 \cdot 3 \cdot 1$ cu. ft. $= 2$ cu. ft.

Volumes of a cylinder and a cone. If a hollow rectangular parallelepiped, a hollow cylinder, and a hollow cone be constructed from pasteboard, all having equal bases and equal altitudes, then, by the method employed for finding the volume of a pyramid, the cylinder will be found to have the same volume as the rectangular parallelepiped, and the cone to have one-third of that volume.

In our work only cones and cylinders, whose bases are circles, will be considered. In the case of right circular cones and cylinders, the curved surfaces are easily measured. For the surface of each may be imagined as unrolled from the solid itself, in which case the surface of the cylinder will unroll into a rectangle, and that of the cone will unroll into a sector of a circle. As a circle has the same area as a rectangle whose base equals the circumference of the circle and whose altitude equals half the radius, so the curved surface of the cone equals a rectangle whose base equals the circumference of the base of the cone and whose altitude equals half the slant height of the cone. Students are advised to construct the figures from pasteboard and to read this paragraph with the solids in hand.

Surface of a sphere. If the surface of a hemisphere be wound by a waxed tape, as in the figure, it will be found to take twice as much tape as is needed to wind a great circle of the sphere. Therefore, the area of the surface equals that of four great circles, or $4\pi r^2$.

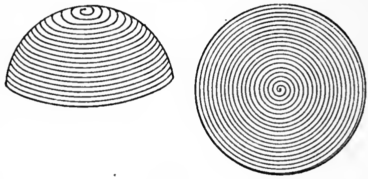


FIG. 14.

E.g., the surface of a sphere whose radius is 5 ft. is $4\pi \cdot 5^2 \cdot 1$ sq. ft. $= 314.16$ sq. ft.

Volume of a sphere. If three points on a sphere be connected with the center, a solid is cut out which is nearly a pyramid with altitude equal to the radius r. If the sphere be divided into any number of such solids, the sum of the bases is the surface of the sphere, $4\pi r^2$, and the common altitude is r. Therefore the volume of all these solids, considered as pyramids, is $\frac{1}{3} \cdot r \cdot 4\pi r^2$, or $\frac{4}{3}\pi r^3$. In geometry it is strictly proved that this is the volume.

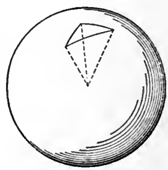


FIG. 15.

E.g., the volume of a sphere whose radius is 5 ft. is $\frac{4}{3}\pi \cdot 5^3 \cdot 1$ cu. ft. $= 523.6$ cu. ft.

Exercises. 1. Find the volume of a pyramid whose base is an equilateral triangle 3 in. on an edge, and whose altitude is 5 in., correct to 0.01 cu. in.

2. What is the volume of the pyramid of Cheops, its base being 764 ft. square and its altitude being 480.75 ft.?

3. What is the weight of a wall 5.40 m long, 2.30 m high, and 0.50 m thick, the wall being entirely composed of stone which is 1.83 times as heavy as water.

4. What is the volume of a cylinder of radius 2 in. and altitude 5 in.?

5. What is the volume of a cone of radius 8 in. and altitude 9 in.?

6. The circumference of a steel shaft is 5 dm; what is the length of its radius?

7. If the shaft in Ex. 6 is 8 m long, what is its volume? What is its weight, steel being 7.8 times as heavy as water?

8. What is the area of the entire surface of a cone whose radius is 5 in. and whose slant height is 4 in.?

9. Find the surfaces of the spheres of radii (*a*) 3 in., (*b*) 15 dm, (*c*) 51 cm, (*d*) 201 mm, (*e*) 415 mm, each correct to 6 significant figures.

10. Find the volumes of the spheres of radii (*a*) 1 ft., (*b*) 6 in., (*c*) 42 cm, (*d*) 66 mm, (*e*) 123 mm.

11. Find the radii of the spheres of surfaces (*a*) 3.1416 sq. in., (*b*) 706.8583 sq. in., (*c*) 5026.5482 m^2, (*d*) 7853.9816 sq. in., (*e*) 70,686 mm^2, each correct to 0.1.

12. Find the radii of the spheres of volume (*a*) 1,436,750 cm^3, (*b*) 523,600 cu. in., (*c*) 195,432 m^3, (*d*) 4,188,800 mm^3, (*e*) 7,979,250 mm^3, correct to units' place.

13. If $s =$ the surface, and $c =$ the circumference, and $v =$ the volume of a sphere of radius r, show that $r = \frac{1}{2}\sqrt{s/\pi}$, $s = c^2/\pi$, $c = \sqrt{\pi s}$, and $r = \sqrt[3]{3\,v/4\,\pi}$.

14. Supposing 1 cu. ft. of water to weigh 1000 oz. and marble to be 2.7 times as heavy as water, find the weight of a sphere of marble 3 ft. in circumference, correct to 0.1 oz.

15. A cylindrical cistern 2 m in diameter is filled with water to the depth of 2 m; what is the weight of the water?

16. A cylindrical tank is 1.20 m in diameter and 3 m long; it is filled with petroleum, which is 0.7 as heavy as water; what is the weight of the petroleum?

17. The diameter of a sphere and the altitude of a cone are equal and they have equal radii; how do their volumes compare?

18. Find the radius of that circle the number of square inches of whose area equals the number of linear inches of its circumference.

19. Find the radius of that sphere the number of square inches of whose surface equals the number of cubic inches of its volume.

20. The radius of the earth being approximately 4000 mi., what is its approximate area, correct to 1000 sq. mi.; also its approximate volume, correct to 1000 cu. mi.

21. From Exs. 14 and 20, compute the weight of the earth to three significant figures, expressing the result in the index notation, knowing that the earth is 5.6 times as heavy as water.

22. What is the weight of a column of water in a pipe 10 m high, of interior diameter 0.09 m?

23. The surface of a pyramid is made up of equilateral triangles 3 in. on a side; find the area.

24. In Ex. 23, given that the perpendicular from the vertex of the pyramid meets the base two-thirds of the way from any of its vertices to the opposite side, find the volume.

25. What is the length of the diagonal of a cube whose edge is 10 in.?

26. What is the length of the edge of a cube whose diagonal is 10 in.?

27. A cube is inscribed in, and another cube is circumscribed about, a sphere whose diameter is 10 in.; find (*a*) their respective volumes, (*b*) the areas of their respective surfaces.

28. What is the cost of 100 km of copper wire 0.53 cm in diameter, at 25 cts. a kilogram, the wire being 8.8 as heavy as water?

29. It is proved in geometry that the volume of a *frustum* of a pyramid or cone (that portion cut off by a plane parallel to the base) is equal to $\frac{1}{3}h(b_1 + b_2 + \sqrt{b_1 b_2})$ cubic units, where $h =$ the *number* of units of height, and b_1, b_2 the *number* of square units in the two bases. Draw the figure and find the volume of a frustum of a pyramid whose bases are squares 6 in. and 8 in. on a side, the altitude being 6 in.

30. Also of a frustum of a cone the radii of whose bases are 3 in. and 5 in., the altitude being 6 in.

31. Also of a frustum of a pyramid whose bases are regular hexagons with sides 4 in. and 6 in., respectively, the altitude being 5 in.

32. From a cube of wood the largest possible sphere is turned; what portion of the cube has been cut away? Answer to 0.001.

CHAPTER VIII.

Longitude and Time.

THE subject of longitude and time is of practical value to two classes of people, navigators and astronomers. While, therefore, it is strictly a part of astronomy or of mathematical geography, custom has assigned to the elements of the subject a place in arithmetic. Since the portions relating to Standard Time and the so-called Date Line belong to the general store of information required by every student, and since the theory forms an interesting application of arithmetic, an entire chapter is assigned to the subject.

The prime meridian, that is, the meridian from which longitude is reckoned, is generally taken as the one passing through Greenwich, England.

In 1675 the Royal observatory of England was erected at Greenwich, a city just below London on the Thames. As the shipping interests of England increased and gradually surpassed those of other nations, the British government was called upon to prepare large numbers of maps, and in time found it more convenient to use the meridian passing through the Royal observatory than the one through the Canary Islands which the ancients had used. This was a signal for several other nations which had continued to use the old prime meridian to use the respective ones passing through their own observatories. In 1884 an International Meridian Congress was held at Washington, and since this meeting most nations, excepting France, have used the Greenwich meridian.

From the prime meridian longitude is reckoned east and west to 180°. West longitude is designated by the symbol + or by the letter W.; east longitude by — or E. If the longitudes of two places are $+45°$ and $+100°$, the difference is $+100° - (+45°) = 55°$; if the longitudes are $-10°$ and $-50°$, the difference is $-10° - (-50°) = 40°$; if the longitudes are $+45°$ and $-50°$, the difference is $+45° - (-50°) = 95°$. In other words, if both longitudes are east, or if both are west, subtract; if one is east and the other west, add.

The two tables of longitude and time. Since the earth revolves upon its axis once in 24 hours, the place in which we live passes through 360° in that length of time. Hence the following tables:

TABLE I.

$\because$ 360° correspond to 24 hrs.
$\therefore$ 1° corresponds to $\frac{1}{360}$ of 24 hrs., or $\frac{1}{15}$ hr., or 4 mins.
$\therefore$ 1′ corresponds to $\frac{1}{60}$ of 4 mins., or $\frac{1}{15}$ min., or 4 secs.
$\therefore$ 1″ corresponds to $\frac{1}{60}$ of 4 secs., or $\frac{1}{15}$ sec.

TABLE II.

$\because$ 24 hrs. correspond to 360°.
$\therefore$ 1 hr. corresponds to $\frac{1}{24}$ of 360°, or 15°.
$\therefore$ 1 min. corresponds to $\frac{1}{60}$ of 15°, or $\frac{1}{4}$°, or 15′.
$\therefore$ 1 sec. corresponds to $\frac{1}{60}$ of 15′, or $\frac{1}{4}$′, or 15″.

The use of these tables may be illustrated by two examples.

Problem. The difference in longitude between two places is 75° 10′ 30″; what is the difference in local time?

Solution. 1. $75 \cdot \frac{1}{15}$ hr. = 5 hrs.
2. $10 \cdot 4$ secs. = 40 secs.
3. $30 \cdot \frac{1}{15}$ sec. = 2 secs.
4. $\therefore$ the difference in time is 5 hrs. 42 secs.

Analysis. $\because$ 1° corr. to $\frac{1}{15}$ hr., 75° corr. to $75 \cdot \frac{1}{15}$ hr., or 5 hrs. Similarly for steps 2, 3. Step 1 might also be stated, $75 \cdot 4$ mins. = 300 mins. = 5 hrs.; and step 2 might be stated, $10 \cdot \frac{1}{15}$ min. = $\frac{2}{3}$ min. = 40 secs. The student should choose the more advantageous multiplicand.

Problem. The difference in local time between two places is 6 hrs. 12 mins. 2 secs.; what is the difference in longitude?

Solution. 1. $6 \cdot 15° = 90°$.
2. $12 \cdot \frac{1}{4}° = 3°$.
3. $2 \cdot 15'' = 30''$.
4. $\therefore$ the difference in longitude is 93° 30″.

Analysis. $\because$ 1 hr. corr. to 15°, 6 hrs. corr. to $6 \cdot 15°$, or 90°. Similarly for steps 2, 3. What other solutions could have been used in steps 2, 3?

Checks. The method of checking either type of problem is evidently found in solving the inverse problem.

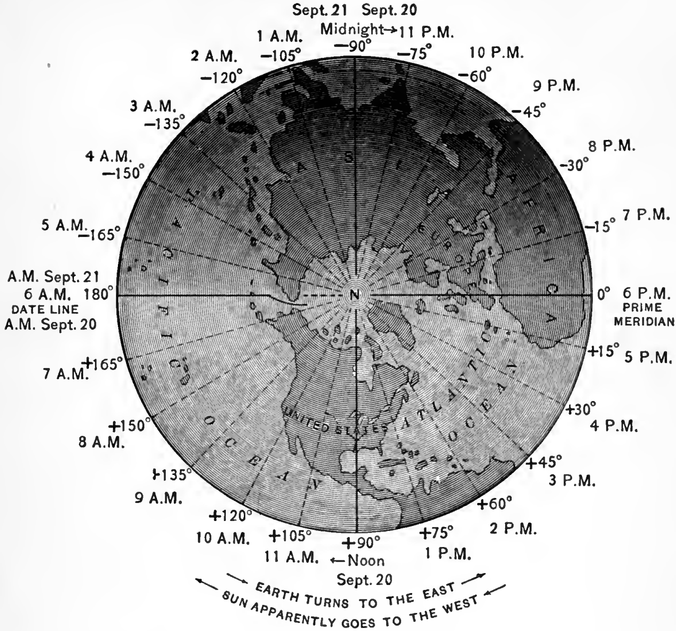
In scientific works the longitude of a place is frequently indicated by quoting the difference in time between that place and Greenwich.

Thus, the longitude of Brussels is given as either − 4° 22′ 9″ or − 17 mins. 28.6 secs.

TABLE OF LONGITUDES FOR REFERENCE IN SOLVING THE SUBSEQUENT PROBLEMS.

Albany, N. Y.	+ 73° 44′ 48″	Honolulu, Hawaii	+ 157° 51′ 48″
Ann Arbor, Mich.	+ 83° 43′ 48″	Jerusalem	− 35° 13′ 25″
Athens, Greece	− 23° 43′ 55.5″	Lisbon, Portugal	+ 9° 11′ 10.5″
Berlin, Germany	− 13° 23′ 43.5″	Madras, India	− 80° 14′ 51″
Berne, Switz.	− 7° 26′ 30″	Madrid, Spain	+ 3° 41′ 21″
Bologna, Italy	− 11° 21′ 9″	Melbourne	− 144° 58′ 42″
Brussels, Belgium	− 4° 22′ 9″	New York	+ 73° 58′ 25.5″
Cairo, Egypt	− 31° 17′ 13.5″	Paris, France	− 2° 20′ 15″
Cambridge, Eng.	− 0° 5′ 40.5″	Peking, China	− 116° 27′ 0″
Cambridge, Mass.	+ 71° 7′ 45″	San Francisco	+ 122° 25′ 40.8″
Cape Town, Africa	− 18° 28′ 45″	Sydney, Australia	− 151° 12′ 39″
Chicago, Ill.	+ 87° 36′ 42″	Tokyo, Japan	− 139° 42′ 30″
Dublin, Ireland	+ 6° 20′ 30″	W'mst'n, Austral.	− 144° 54′ 42″

Exercise. What is the difference between the longitudes and between the local times of: (*a*) Berne and Chicago? (*b*) Dublin and San Francisco? (*c*) Melbourne and Madrid? (*d*) New York and Cambridge, Eng.? (*e*) Tokyo and Cape Town? (*f*) Ann Arbor and Berlin? (*g*) Bologna and Melbourne? (*h*) Brussels and Jerusalem? (*i*) Cairo and Williamstown? (*j*) Lisbon and Madras?



The above figure represents the earth when it is noon Sept. 20, on the meridian $+90°$, as seen from a point above the north pole. It is, therefore, midnight directly opposite, that is, at $-90°$, and it is A.M. to the left of the noon line and P.M. to the right. The student should now consider the following questions:

1. In the figure, how do we know that it is P.M. in New York?
2. Also A.M. in California, Alaska, Hawaii, Japan?
3. If it is 11 P.M. Sept. 20 at $-75°$, what *date* is it on the other side of midnight, at $-105°$?
4. Hence, what *date* is it in Japan?
5. What date is it in California?
6. Hence, there must be a line in the Pacific ocean which separates what two dates?

The answers to the questions on p. 83 show the necessity for a line at which Sept. 21, and in general each new day, shall begin on the earth. This line is, by common consent, usually taken as nearly coinciding with the 180° meridian, ships changing their calendars one day on crossing this line.

Exercises. 1. Draw a map similar to that on p. 83, but without details (*a*) showing the arrangement of days and hours when it is noon Nov. 25 at $-15°$; (*b*) showing the same for $-90°$; (*c*) also for 0°; (*d*) also for 180°; (*e*) also for $-150°$; (*f*) also for $+150°$. (Separate map for each case.)

2. When it is noon, local time, Jan. 1, 1900, at San Francisco, what is the date and the local time at (*a*) New York? (*b*) Greenwich? (*c*) Cape Town? (*d*) Melbourne? Draw a map, without details, illustrating the problem.

3. When it is Sunday noon, local time, at Sydney, what is the day and the local time at (*a*) Cairo? (*b*) Dublin? (*c*) Chicago? (*d*) Honolulu? (*e*) Peking? (*f*) Greenwich? Draw a map, without details, illustrating the problem.

4. What is the longitude of that place at which, when it is noon at Greenwich, it is (*a*) 6 P.M.? (*b*) 2 A.M.? (*c*) 5 o'clock 20 min. A.M.? (*d*) 7 o'clock 50 min. 10 sec. P.M.?

5. What is the difference between the time of Greenwich and the local time of: (*a*) Albany? (*b*) Athens? (*c*) Cambridge, Eng.? (*d*) Honolulu? (*e*) Melbourne? (*f*) Madras? (*g*) Jerusalem? (*h*) Chicago?

6. What is the difference between the longitudes and between the local times of: (*a*) Williamstown and Honolulu? (*b*) Peking and San Francisco? (*c*) New York and Sydney? (*d*) Melbourne and Chicago?

7. When it is noon local time at Chicago, on what meridian is it midnight?

8. A ship at sea finds that it is 3 hrs. 34 mins. P.M. by its Greenwich chronometer when the sun is on the meridian; what is the ship's longitude?

It is suggested that the teacher ascertain the approximate longitude of the place in which he is teaching and add this to the table. The number of problems which can be made from the table is limitless, the above being merely types.

Standard time. In 1883 the railways of the United States and Canada proposed a system of uniform time which has since been quite generally adopted, not only in this country, but throughout the whole civilized world. In the United States and Canada that section of country lying about $7\frac{1}{2}°$ east and west of $+75°$ uses the time of $+75°$; that section lying about $7\frac{1}{2}°$ on either side of $+90°$ uses the time of $+90°$; and similarly for the meridians of $+105°$, $+120°$, and $+60°$. Since the movement was primarily for the accommodation of the railways, the lines of division are not exactly $7\frac{1}{2}°$ on either side of the hour meridians, but are usually passed through the leading railway termini in the vicinity.

Exercises. (The following exercises refer to standard time only.)

1. When it is noon at New York, what is the time at Chicago, San Francisco, Denver, New Orleans, Boston ?

2. When it is 11 hrs. 30 mins. P.M. at San Francisco, what is the time at Denver, Milwaukee, Detroit, Albany, Philadelphia ?

3. When it is 1 o'clock 15 mins. A.M. at Boston, what is the time at New Haven, Cincinnati, St. Louis, Portland, Oregon, Colorado Springs ?

At present, the following are some of the countries using standard time based on the hour meridians (multiples of 15°) from Greenwich, but the movement is so recent that the list is not intended to be complete.

0°, or west European time: Great Britain, Holland, Belgium.

— 15°, or mid-European time: Norway, Sweden, Denmark, Germany, Austria, Switzerland, Italy, Servia, Western Turkish railways.

— 30°, or east European time: Bulgaria, Roumania, railways to Constantinople. Also Natal.

— 120°: western Australia.

— 135°: southern Australia, Japan.

— 150°: eastern Australia.

France and Algiers use uniform time, that of Paris. Cape Colony (South Africa) uses $-22\frac{1}{2}°$.

The following exercises refer to *standard time*, except as otherwise stated.

Exercises. 1. When it is 9 A.M. Nov. 20 at Chicago, what is the date and time at London, Philadelphia, San Francisco, New Orleans, Montreal, Denver, Melbourne (— 150° time), Venice, Constantinople (— 30° time)?

2. When it is 11 P.M. Dec. 31, 1899, at New York, what is the date and time at Denver, San Francisco, St. Louis, Boston, London, Berlin, Tokyo?

3. When it is Sunday noon at San Francisco, what is the day and time at Tokyo, Williamstown (— 150° time), Rome, Amsterdam, New York?

4. When would a press telegram sent from Berlin at 2 A.M. Jan. 1 reach Portland, Oregon, if transmitted without any delay? When would one sent from Melbourne at 2 A.M. Jan. 1 reach San Francisco, if transmitted without delay?

5. What is the difference between the local and standard time at each of the following places: (*a*) New York? (*b*) Chicago? (*c*) Cape Town? (*d*) Cambridge, Mass.? (*e*) Brussels? (*f*) Berne? (*g*) Berlin?

6. At what hour (standard time) does your arithmetic class meet? What is, then, the time at Glasgow, The Hague, Stockholm, Lucerne, Naples, Melbourne, Yokohama, Cape Town?

7. Twenty-four hour clocks (by which 1 P.M. is 13 o'clock, etc.) are used in certain parts of the world; when it is 18 o'clock in Italy, what time is it in Belgium? at Chicago?

CHAPTER IX.

Ratio and Proportion.

I. Ratio.

The *ratio* of one number, a, to another number, b, of the same kind, is the quotient $\frac{a}{b}$.

Thus, the ratio of \$2 to \$5 is $\frac{\$2}{\$5}$, or $\frac{2}{5}$, or 0.4,

and the ratio of 4 ft. to 2 ft. is $\frac{4 \text{ ft.}}{2 \text{ ft.}}$, or $\frac{4}{2}$, or 2.

But there is no ratio of 4 m to 3, or \$5 to 2 ft., or 2 to \$10.

A ratio may be expressed by any symbol of division, *e.g.*, by the fractional form, by $\div$, by $/$, or by $:$; but the symbols generally used are the fraction and the colon, as $\frac{a}{b}$ or $a:b$.

The ratio $\frac{b}{a}$ is called the *inverse* of the ratio $\frac{a}{b}$.

If two variable quantities, a, b, have a constant ratio r, one is said to *vary as* the other.

E.g., the ratio of any circumference to its diameter is π; hence, a circumference is said to vary as its diameter.

If $\frac{a}{b} = r$, then $a = rb$. The expression "a varies as b" is sometimes written $a \propto b$, meaning that $a = rb$.

If $a = r \cdot \frac{1}{b}$, a is said to *vary inversely* as b.

If two variable quantities, a, b, have the same ratio as two other variable quantities, a', b', then a and b are said to *vary as* a' and b'. And if any two values of one variable quantity have the same ratio as the corresponding values of another variable quantity which depends on the first, then one of these quantities is said to vary as the other.

E.g., the circumference c and diameter d of one circle have the same ratio as the circumference c' and diameter d' of any other circle; hence, c and d are said to vary as c' and d'.

If two rectangles have the same altitude, their areas depend on their bases; and since any two values of their bases have the same ratio as the corresponding values of their areas, their areas are said to vary as their bases.

The theory of ratio has its applications in geometry, in physics, and in practical business.

Applications in geometry. Similar figures may be described as figures having the same shape, such as lines, squares, triangles whose angles are respectively equal, circles, cubes, or spheres. It is proved in geometry that in two similar figures

(1) *Any two corresponding lines vary as any other two corresponding lines.*

(2) *Corresponding areas vary as the squares of any two corresponding lines.*

(3) *Corresponding volumes vary as the cubes of any two corresponding lines.*

E.g., in the case of two spheres, the circumferences vary as the radii, the surfaces vary as the squares of the radii, the volumes vary as the cubes of the radii.

These facts are easily proved. Let s, s' stand for the surfaces of two spheres of radii r, r', respectively. Then,

$$s = 4\,\pi r^2, \text{ and } s' = 4\,\pi r'^2,$$

$$\therefore \frac{s}{s'} = \frac{4\,\pi r^2}{4\,\pi r'^2} = \frac{r^2}{r'^2}.$$

Hence, the surfaces vary as the squares of the radii. In like manner the volumes might be considered.

Exercises. 1. The ratio of 2 to x is 5; find x.

2. Find x in the following ratios: (*a*) $\frac{x}{3} = 7$, (*b*) $4 : x = 9$, (*c*) $x : 17 = 10$, (*d*) $\frac{36}{x} = x$, (*e*) $x = \frac{1}{144} : x$.

3. Find x in the following ratios: (*a*) $\frac{x}{5} = 2.4$, (*b*) $7 : x = 4.9$, (*c*) $x : 5 = \frac{3}{4}$, (*d*) $\frac{3}{x} = \frac{9}{8}$, (*e*) $\frac{x}{2} = \frac{3}{7}$.

4. The surfaces of a certain sphere and a certain cube have the same area; find, to 0.01, the ratio of their volumes.

5. In drawing a circle of radius 100 ft. on a scale of $\frac{1}{1200}$, what length would represent the side of a square inscribed in the circle?

6. If the distance between two cities 256 mi. apart appears on a map as 2.56 in., what is the scale on which the map is drawn?

7. One cube is 1.2 times as high as another; find the ratio of (*a*) their surfaces, (*b*) their volumes.

8. At that time of the day when the length of a man's shadow is $\frac{3}{4}$ of his height, the shadow of a telegraph pole was found to be 27.8 ft.; find the height of the pole.

9. If a sphere of lead weighs 4 lbs., find the weight of a sphere of lead of (*a*) twice the volume, (*b*) twice the surface, (*c*) twice the radius.

10. Explain Newton's definition of number: Number is the abstract ratio of one quantity to another of the same kind. What kinds of numbers are represented in the following cases: 5 ft. : 1 ft., 1 ft. : 5 ft., the diagonal to the side of a square, the circumference to the diameter of a circle?

11. Two lines are respectively 7.9 m and 23.7 m long; what is the ratio of the first to the second? the second to the first?

12. Two arcs of the same circumference are respectively $85° 31' 22''$ and $30° 17' 27.7''$; express the ratio of the first to the second, correct to 0.001.

13. The equatorial radius of the earth is 6,377,398 m, and the polar radius 6,356,080; find the ratio of their difference to the former, correct to 0.01.

14. The depths of three artesian wells are as follows: A 220 m, B 395 m, C 543 m; the temperatures of the water from these depths are: A 19.75° C., B 25.33° C., C 30.50° C. From these observations is it correct to say that the increase of temperature is proportional to the increase of depth? If not, what should be the temperature at C to have this law hold?

Applications in business. Of the numerous applications of ratio in business, only a few can be mentioned, and not all of these commonly make use of the word "ratio."

In computing interest, the simple interest varies as the time, if the rate is constant; as the rate, if the time is constant; as the product of the rate and the number representing the time in years (if the rate is by the year), if neither is constant.

I.e., for twice the rate, the interest is twice as much, if the time is constant; for twice the time, the interest is twice as much, if the rate is constant; but for twice the time and 1.5 times the rate, the interest is $2 \cdot 1.5$ as much.

The common expressions "2 out of 3," "9 out of 10," "2 to 5," "6 per cent" (merely 6 out of 100) are only other methods of stating the following ratios of a part to a whole, $\frac{2}{3}$, $\frac{9}{10}$, $\frac{2}{7}$, $\frac{6}{100}$, or the following ratios of the two parts, $\frac{2}{1}$, $\frac{9}{1}$, $\frac{2}{5}$, $\frac{6}{94}$.

E.g., to divide \$100 so that A shall receive \$2 out of every \$3 is to divide it into two parts

(*a*) having the ratio 2 : 1, or

(*b*) so that A's share shall have to the whole the ratio 2 : 3, or

(*c*) so that B's share shall have to the whole the ratio 1 : 3.

To divide \$1000 in the ratio of 7 : 8.

1. There are 15 parts of which 7 are to go into one share.

2. $\therefore \frac{x}{\$1000} = \frac{7}{15}$; and $\because$ the dividend x is the product of the divisor \$1000 and the quotient $\frac{7}{15}$,

3. $\therefore x = \frac{7}{15}$ of \$1000, or \$466.67.

4. $\therefore$ the other share is \$1000 − \$466.67 = \$533.33.

Other applications will be seen in the following exercises.

Exercises. 1. Divide \$1000 so that A shall have \$7 out of every \$8.

2. Divide \$500 between A and B so that A shall have \$0.25 as often as B has \$1.25.

3. The area of the United States is 3,501,000 sq. mi., and the area of Russia is 8,644,100 sq. mi.; express the ratio of the former to the latter as a fraction with the denominator 100.

4. From the following table of German statistics find, to 0.01, the ratios of the numbers of crimes for each pair of years, and also of the prices of grain for the same periods.

Date.	Crimes.	Price of Grain.
1882	535	152.3 marks per kg.
1885	486	140.6 "
1888	459	134.5 "
1891	511	211.2 "

5. The following table gives statistics of workers in the woolen trade in 1895. Find the ratios of the expenditures for each of the three purposes, to the incomes, in each country, correct to 0.001.

	Income.	Expenditures.		
		Tobacco.	Intoxicants.	Religion.
United States	$663.13	$9.36	$18.39	$8.37
Great Britain	515.64	9.07	16.01	6.34
France	424.51	7.01	33.72	3.25
Germany	275.99	3.08	11.74	1.19

6. The white population of the United States in 1780 was 2,383,000; in 1790, 3,177,257; in 1880, 43,402,970; in 1890, 54,983,890. What is the ratio of the population in 1790 to that in 1780? in 1890 to that in 1880? (Each correct to 0.01.)

7. Before the city of Munich had sewers and an abundant supply of pure water, the annual death rate from typhoid fever was 242 out of 100,000; after these improvements the rate sank to 14 out of 100,000. If the death rate from this disease in a town without these improvements is 18 out of 10,000, what would it be with the improvements and with the same rate of decrease as in Munich?

8. The number of women employed in the United States was

	1870.	1890.		1870.	1890.
in art	412	10,810	in music	5,753	34,519
as authors	159	2,725	as stenographers	7	21,185
as journalists	35	88	as teachers	84,047	245,965

Find, correct to 0.1, the ratios of those employed in the several branches in 1890 to those in 1870.

9. In 1860, out of the $316,242,423 of our exports, $40,345,892 represented manufactured products; in 1890, out of $845,293,828, $151,102,376 represented manufactured products. Find, correct to 0.01, the ratio of the manufactured to the total in each year.

Applications in physics. (*a*) *Specific gravity.* The specific gravity of any substance is the ratio of the weight of that substance to the weight of an equal volume of some other substance taken as a standard.

In the case of solids and liquids, distilled water is usually taken as the standard. Thus, the specific gravity of mercury, of which 1 l weighs 13.596 kg, is 13.596, because this is the ratio of the weight of a liter of the substance to the weight of a liter of water;

$$i.e., \ 13.596 \text{ kg} : 1 \text{ kg} = 13.596.$$

In the case of gases, either hydrogen or air is usually taken as the standard.

The following table will be needed for reference in solving the exercises.

SPECIFIC GRAVITIES, REFERRED TO WATER.

Copper	8.9.	Nickel	8.9.	Cork	0.24.	Alcohol	0.79.
Gold	19.3.	Silver	10.5.	Granite	2.7.	Petroleum	0.7.
Lead	11.3.	Sulphur	2.0.	Steel	7.8.	Mercury	13.596.

SPECIFIC GRAVITIES, REFERRED TO HYDROGEN.

Air	14.43.	Oxygen	15.95.	Coal gas	6.

SPECIFIC GRAVITIES, REFERRED TO AIR.

Oxygen	1.11.	Hydrogen	0.07.	Chlorine gas	2.44.

WEIGHTS OF CERTAIN SUBSTANCES.

1 l of water, 1 kg.	1 l of air, 1.293 g.
1 cm^3 of water, 1 g.	1 cu. ft. of water, about 62.5 lbs., or about 1000 oz.

Example. What is the weight of 1 cu. in. of copper?

1. 1 cu. ft. of water weighs 1000 oz.
2. $\therefore$ 1 cu. in. of water weighs $\dfrac{1000 \text{ oz.}}{1728}$
3. $\therefore$ 1 cu. in. of copper weighs $\dfrac{8.9 \cdot 1000 \text{ oz.}}{1728} = 5.15$ oz.

Exercises. 1. What is the weight of a cubic foot of copper? of gold? of lead?

2. What is the weight of 1 cm^3 of nickel? of silver? of sulphur?

3. What is the weight of 1 dm^3 of cork? of granite? of steel?

4. What is the weight of 1 l of alcohol? of petroleum? of mercury?

5. How could the theory of specific gravity be applied to finding the purity of milk, the average specific gravity of good milk being 1.032?

6. What is the specific gravity of air, referred to air? referred to hydrogen? referred to water?

7. What is the specific gravity of coal gas, referred to air?

8. Sodium has a specific gravity of 1.23, referred to alcohol; what is its specific gravity, referred to water?

9. What is the weight of 1 dm^3 of oxygen? of hydrogen?

10. A cubic foot of green oak weighs 73 lbs., of iron 432 lbs.; find the specific gravity of each, correct to 0.01.

11. An empty balloon weighs 1200 lbs.; if 1 cu. ft. of air weighs 1.25 oz., how many cubic feet of gas, of specific gravity 0.52 with respect to air, must be introduced before the balloon will begin to ascend?

12. The specific gravity of sea-water is 1.026; how many cubic feet weigh one ton?

13. A flask holds 27 oz. of water; what is the weight of alcohol that it will hold? of petroleum? of mercury?

14. In a liter jar are placed 1 kg of lead and 1 kg of copper; what volume of water is necessary to fill the jar?

15. A nugget of gold mixed with quartz weighs 0.5 kg; the specific gravity of the nugget is 6.5, and of quartz 2.15; how many grams of gold in the nugget?

16. A vessel containing 1 l and weighing 0.5 kg is filled with mercury and water; it then weighs, with its contents, 3 kg; how many cm^3 of each in the vessel?

17. The specific gravity of a certain liquid is 2.000 at 0°, 1.950 at 10°, 1.300 at 100°; find the volume of 100 g of the liquid at each of these temperatures.

18. A cylindrical vessel 1 dm in diameter is filled with mercury to the height of 8 cm; what is the pressure, in grams, upon the base?

19. What must be the height of a column of mercury to exert a pressure of 0.5 kg per cm^2?

20. The specific gravity of ice is 0.92, of sea-water 1.025; to what depth will a cubic foot of ice sink in sea-water?

21. From Ex. 20, how much of an iceberg 500 ft. high would show above water, the cross section being supposed to have a constant area?

(*b*) *Law of levers.* If a bar AB rests on a fulcrum F and has a weight w at B, then by exerting enough pressure p at A the weight can be raised. In the first figure the pressure is downward (positive pressure); in the second it is upward (negative pressure).

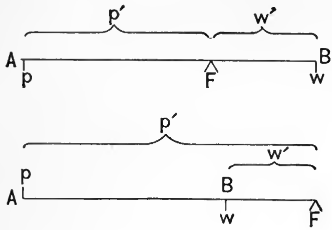


FIG. 17.

There is a law in physics that, if p′, w′ represent the number of units of distance AF, FB, respectively, and p, w the number of units of pressure and weight, respectively, then $\frac{pp'}{ww'} = 1$.

In the first figure p, w, p′, w′ are all considered as positive; in the second figure p is considered as negative because the pressure is upward, and w′ is considered as negative because it extends the other way from F. Hence, the ratio $pp' : ww' = 1$ in both cases.

Example. Suppose AF = 25 in., FB = 14 in., in the first figure; what pressure must be applied at A to raise a weight of 30 lbs. at B?

1. By the law of levers $\frac{25\ p}{14 \cdot 30} = 1$.
2. $\therefore p = \frac{14 \cdot 30}{25} = 16.8$, and $\therefore$ the pressure must be 16.8 lbs.

Exercises. 1. Two bodies weighing 20 lbs. and 4 lbs. balance at the ends of a lever 2 ft. ong; find the position of the fulcrum.

2. The radii of a wheel and axle are respectively 4 ft. and 6 in.; what force will just raise a mass of 56 lbs., friction not considered?

3. What pressure must be exerted at the edge of a door to counteract an opposite pressure of 100 lbs. halfway from the hinge to the edge? one-third of the way from the hinge to the edge?

4. The length of the spoke of a capstan is 6 ft. measured from the axis, and the radius of the drum is 1 ft.; find the weight of an anchor that can just be raised by 6 men, each exerting a force equal to 100 lbs. at the end of a spoke, friction not considered.

5. In each figure, what must be the distance AF in order that a pressure of 1 kg may raise a weight of 100 kg 3 dm from F?

(*c*) *Boyle's law.* It is proved in physics that if p is the number of units of pressure of a given quantity of gas, and v is the number of units of volume, then p varies inversely as v when the temperature remains constant.

This law was discovered in the seventeenth century by Robert Boyle.

E.g., if the volume of a gas is 10 dm³ under the ordinary pressure of the atmosphere ("under a pressure of one atmosphere"), it is

$\frac{1}{n}$ as much when the pressure is n times as great,

n times " " " " " " $\frac{1}{n}$ "

the temperature always being considered constant.

Example. A toy balloon contains 3 l of gas when exposed to a pressure of 1 atmosphere; what is its volume when the pressure is increased to 4 atmospheres? decreased to $\frac{1}{8}$ of an atmosphere?

1. $\because$ the volume varies inversely as the pressure, it is $\frac{1}{4}$ as much when the pressure is 4 times as great.

2. Similarly, it is 8 times as much when the pressure is $\frac{1}{8}$ as great.

3. $\therefore$ the volumes are 0.75 l and 24 l.

Exercises. 1. Equal quantities of air are on opposite sides of a piston in a cylinder that is 12 in. long; if the piston moves 3 in. from the center, find the ratios of the pressures. Draw the figure.

2. If a cylinder of gas under a certain pressure has its volume increased from 20 l to 25 l, what is the ratio of the pressures?

3. A cubic foot of air weighs 570 gr. at a pressure of 15 lbs. to the square inch; what will a cubic foot weigh at a pressure of 10 lbs.?

4. A mass of air occupies 18 cu. in. under a pressure of $7\frac{1}{2}$ lbs. to the square inch; what space will it occupy under a pressure of 25 lbs.?

5. If the volume of a gas varies inversely as the height of the mercury in a barometer, and if a certain mass occupies 23 cu. in. when the barometer indicates 29.3 in., what will it occupy when the barometer indicates 30.7 in.?

6. A liter of air under ordinary pressure weighs 1.293 g when the barometer stands at 76 cm; find the weight when the barometer stands at 82 cm, the weight varying inversely as the height of the barometer.

7. A certain gas has a volume of 1200 cm³ under a pressure of 1033 g to 1 cm²; find the volume when the pressure is 1250 g.

II. Proportion.

A *proportion* is an expression of equality of ratios.

Thus, $\frac{2}{3}=\frac{4}{6}$, $\frac{\$7}{\$8}=\frac{14\text{ men}}{16\text{ men}}$, $\$3.50:\$7=4$ books : 8 books, $3/4=\$6/\$8$, are examples of proportion. There is another symbol formerly much used to express the equality of ratios, the double colon (: :).

There may be an equality of several ratios, as $1:2=4:8=9:18$, the term *continued proportion* being applied to such an expression. There may also be an equality between the products of ratios, as $\frac{2}{3}\cdot\frac{5}{7}=\frac{1}{7}\cdot\frac{10}{3}$, such an expression being called a *compound proportion.*

Arithmetic uses continued proportion but little, and problems in compound proportion are more easily solved by the unitary analysis.

In the proportion $a:b=c:d$, a, b, c, d are called the *terms*, a and d the *extremes*, and b and c the *means.*

If three of the four terms of a proportion are known, the other can be readily found by multiplying or dividing equals by equals. Thus, if $\frac{x}{7}=\frac{4}{8}$, then $x=\frac{7\cdot 4}{8}=3.5$. If this should seem to require multiplying by a concrete number, the difficulty may be avoided by making each term of either ratio abstract; this is permissible because $\frac{\$4}{\$3}=\frac{4}{3}$.

Exercises. 1. Given $\frac{a}{b}=\frac{c}{d}$, the terms being abstract numbers, to prove that the product of the means equals the product of the extremes.

2. In the proportion $\frac{x}{b}=\frac{c}{d}$, prove that $x=bc/d$; that is, that one extreme equals the product of the means divided by the other extreme.

3. In which of these proportions is the value of x the more easily found, and why? $\frac{x}{4}=\frac{17}{19}$, $\frac{23}{10}=\frac{5}{x}$.

4. Find the value of x, correct to 0.1, in the proportions $\frac{x}{2}=\frac{7}{19}$, $x:3=4:5$, $0.3:7=x:1.52$, $\frac{2}{0.5}=\frac{x}{7}$.

5. Also in the proportions $x:1.273=0.4:2.3$, $1.7:3=x:7$.

If one quantity varies directly as another, the two are said to be *directly proportional,* or simply *proportional.*

E.g., at retail the cost of a given quality of sugar varies directly as the weight; the cost is then proportional to the weight. Thus, at 4 cts. a pound 12 lbs. cost 48 cts., and 4 cts. : 48 cts. = 1 lb. : 12 lbs.

If one quantity varies inversely as another, the two are said to be *inversely proportional.*

E.g., in general, the temperature being constant, the volume of a gas varies inversely as the pressure, and the volume is therefore said to be inversely proportional to the pressure.

Exercises. 1. State which of the following, other things being equal, are directly proportional and which are inversely proportional:

(*a*) Volume of gas, pressure.
(*b*) Volume of gas, temperature.
(*c*) Distance from fulcrum, weight.
(*d*) Cost of carrying goods, distance.
(*e*) Weight of goods carried for a given sum, distance.
(*f*) Amount of work done, number of workers.
(*g*) Price of bread, price of wheat.

2. Given $1.43 : x = 4.01 : 2$, find, correct to 0.01, the value of x.

3. Also in $\frac{x}{7} = \frac{63}{x}$.

4. Also in $27 : x = x : 48$.

The applications of proportion are found chiefly in geometry and physics. While several types of business problems were formerly solved by this means, other methods are now generally employed.

In the two illustrative examples on p. 98, the first three steps are explanatory of the statement of the proportion and may be omitted in practice. In the first problem the ratios are written in the fractional form in order that the reasons involved may appear more readily. The symbol for the unknown quantity may be placed in any term and the proportion arranged accordingly; but if the solution is to be explained, it will be found more convenient to place the x in the first term. (See p. 96, Ex. 3.)

Examples. (*a*) The time of oscillation of a pendulum is proportional to the square root of the number representing its length; the length of a 1-sec. pendulum being 39.2 in., what is the length of a 2-sec. pendulum?

1. Let $x =$ the *number* of inches of length.
2. Then $\frac{x}{39.2} =$ the ratio of the lengths.
3. And $\frac{2}{1} =$ the ratio of the corresponding times of oscillations.
4. $\because$ the time is proportional to the square root of the number representing the length,

$$\therefore \frac{\sqrt{x}}{\sqrt{39.2}} = \frac{2}{1}.$$

5. $\therefore \frac{x}{39.2} = \frac{4}{1}$, whence $x = 39.2 \cdot 4 = 156.8$.
6. $\because x =$ the number of inches, $\therefore$ the pendulum is 156.8 in. long.

(*b*) A mass of air fills 10 dm^3 under a pressure of 3 kg to 1 cm^2; what is the space occupied under a pressure of 5 kg to 1 cm^2, the temperature remaining constant? (See Boyle's law, p. 95.)

1. Let $x =$ the *number* of dm^3 under a pressure of 5 kg to 1 cm^2.
2. Then $x : 10 =$ the ratio of the volumes.
3. And $5 : 3 =$ the ratio of the corresponding pressures.
4. $\because$ the volume is inversely proportional to the pressure,

$$\therefore x : 10 = 3 : 5.$$

5. $\therefore x = 10 \cdot 3 : 5 = 6$.
6. $\because x =$ the number of dm^3, $\therefore$ the space is 6 dm^3.

Exercises. 1. The distance through which a body falls from a state of rest is proportional to the square of the number representing the time of fall; if a body falls 176.5 m in 6 secs., how far does it fall in 3.25 secs.?

2. Also in 1 sec.? in 2 secs.? in 4 secs.?

3. It is proved in mechanics that, neglecting friction, the power acting parallel to an inclined plane and necessary to support a weight is to that weight as the height to which the weight is raised is to the length of the incline. (In the figure, $p : w = h : l = h' : l'$.) If the height is $\frac{3}{4}$ of the length, what power will support a 20-lb. weight (neglecting friction in this and similar problems)?

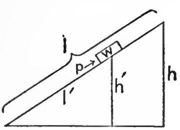


FIG. 18.

4. A train weighing 126 tons rests on an incline and is kept from moving down by a force of 1500 lbs.; the road rises 1 ft. in how many feet of its length?

5. On a plane rising 3 ft. in every 5 ft. of its length, how many pounds of force exerted parallel to the plane will keep a mass of 10 lbs. from sliding?

6. What must be the length of an inclined plane in order that a man may roll a 500-lb. cask into a wagon 3.5 ft. high by the exertion of a force of 350 lbs.?

7. How long is a pendulum which oscillates 56 times a minute?

8. If a pipe 1.5 cm in diameter fills a reservoir in 3.25 mins., how long will it take a pipe 3 cm in diameter to fill it?

9. If a projectile 8.1 in. in length weighs 108 lbs., what is the weight of a similar projectile 9.37 in. long? (Answer to 0.1 lb.)

10. The masses of two solids remaining unchanged, their attraction for each other is inversely proportional to the square of the number representing the distance between their centers of gravity; if at a distance of 2 m they are attracted by a force of 1 mg, what will be their attraction at a distance of 1 km?

11. A body weighs 15 lbs. 5000 mi. from the earth's center (*i.e.*, about 1000 mi. above the surface); how much will it weigh 4000 mi. from the center?

12. If a body weighs 1 kg at the level of the sea, how much will it weigh at an elevation of 1 km? (Take the radius as 6370 km; give the result correct to 0.001 kg.)

13. What is the height of a tower which casts a shadow 143 ft. long at the same time that a post 4.5 ft. high casts a shadow 6.1 ft. long?

14. Of two bottles of similar shape one is twice as high as the other; the smaller holds 0.5 pt., how much does the larger hold?

15. The amount of light received on a given surface being inversely proportional to the square of the number representing its distance from the source of light, if the amount per sq. in. at a distance of 1 ft. is represented by 1, what will represent the amount at a distance of 0.1 in.? (Answer to 0.01.)

16. If the interest received on a certain sum for 1.5 yrs. is $27.50, how much is the interest on the same sum at the same rate for 2 mo.?

17. If the interest received on a certain sum for a certain time is $53 at 6 cts. for every dollar, how much is the interest on that sum for the same time at $4\frac{1}{2}$ cts. for every dollar?

18. When the barometer stands at 30 in., the pressure of the atmosphere is 14.7 lbs. per sq. in.; what is the pressure when the barometer stands at 28 in.?

19. What is the width of a stream if a pole 64 ft. high 9 ft. from its bank casts a shadow which just reaches across the stream and the shadow of a nail in the pole 8 ft. from the ground just touches the bank?

20. A water tower 160 ft. high and 35 ft. in diameter is to be represented in a drawing as 10 in. high; how many inches in the representation of the diameter?

21. The Washington monument casts a shadow 223 ft. 6.5 in. when a post 3 ft. high casts a shadow 14.5 in.; find the height of the monument.

22. If a triangle whose base is 2 in. long has an area of 3 sq. in., what is the area of a similar triangle whose corresponding base is 8 in. long?

23. If a metal sphere 10 in. in diameter weighs 327.5 lbs., what is the weight of a sphere of the same substance 14 in. in diameter?

24. A cube of water 1.8 dm on an edge weighs how many kg?

25. If a sphere whose surface is 16π cm^2 weighs 5 kg, what is the weight of a sphere of the same substance whose surface is 32π cm^2?

26. If the length of a 1-sec. pendulum be considered as 1 m, what is the time of oscillation of a pendulum 6.4 m long? 62.5 m long?

27. On a map constructed on a scale of $\frac{1}{160000}$ the distance from Detroit to Chicago is 112.86 in.; how many miles between these cities?

28. The ratio of immigrants from the United Kingdom to those from the rest of Europe during the decade from 1881 was 5 : 16; the total number from these two sources was 6,192,000; how many from each? (Answer correct to 1000.)

29. Kepler showed that the squares of the numbers representing the times of revolution of the planets about the sun are proportional to the cubes of the numbers representing their distances from the sun. Mars being 1.52369 as far as the earth from the sun, and the time of revolution of the earth being 365.256 da., find the time of revolution of Mars.

30. Similarly, find the time of revolution of each of the following planets, the numbers representing relative distances from the sun, the distance to the earth being taken as the unit: Mercury 0.39, Venus 0.72, Jupiter 5.20, Saturn 9.54, Uranus 19.18, Neptune 30.07.

Problems in electricity. The great advance in electricity in recent years renders necessary a knowledge of such technical terms as are in everyday use. Problems involving these terms belong to proportion, but may be omitted without interfering with the subsequent work.

When water flows through a pipe some *resistance* is offered due to friction or other impediment to the flow of the water.

When electricity flows through a wire some *resistance* is offered. This *resistance is measured in ohms.* An *ohm* is the resistance offered by a column of mercury 1 mm² in cross section, 106 cm long, at 0° C.

A certain *quantity* of water flows through the pipe in a second, and this may be stated in gallons or cubic inches, etc.

A certain *quantity* of electricity flows through the wire. This *quantity is measured in amperes.* An *ampere* is the current necessary to deposit 0.001118 g of silver a second in passing through a certain solution of nitrate of silver.

A certain *pressure* is necessary to force the water through the pipe. This pressure may be measured in pounds per sq. in., kilograms per cm², etc.

A certain *pressure* is necessary to force the electricity through the wire. This *pressure is measured in volts.* A *volt* is the pressure necessary to force 1 ampere through 1 ohm of resistance.

Hence, in considering the water necessary to do a certain amount of work (as to turn a water-wheel) it is necessary to consider not merely the *pressure*, for a little water may come from a great height, nor merely the *volume*, nor merely the *resistance* of the pipe; all three must be considered.

Hence, in considering the electricity necessary to do certain work it is necessary to consider not merely the *voltage*, for a little electricity may come with a high pressure, nor merely the *amperage*, nor merely the *number of ohms of resistance*; all three must be considered.

The names of the electrical units mentioned come from the names of three eminent electricians, Ohm, Ampère, and Volta.

It is proved in physics that the resistance of a wire varies directly as its length and inversely as the area of a cross section.

That is, if a mile of a certain wire has a resistance of 3.58 ohms, 2 mi. of that wire will have a resistance of $2 \cdot 3.58$ ohms, or 7.16 ohms. Also, 1 mi. of wire of the same material but of twice the sectional area will have a resistance of $\frac{1}{2}$ of 3.58 ohms, or 1.79 ohms.

From these laws and definitions, the most common problems and statements concerning electrical measurements will be understood. The student should not feel obliged, however, to use the proportion form in the solutions. Ordinary analysis, the unitary analysis, or the equation may be employed.

Exercises. 1. If the resistance of 1 mi. of a certain electric-light wire is 3.58 ohms, what is the resistance of 5 mi. of wire of the same material but of twice the sectional area? Also of 1 mi. of wire of the same material but of twice the diameter?

2. If the resistance of 700 yds. of a certain cable is 0.91 ohm, what is the resistance of 1 mi. of that cable?

3. If the resistance of 100 yds. of a certain wire is 5 ohms, what length of the same wire would have a resistance of 13.2 ohms?

4. The resistance of a certain wire is 9.1 ohms, and the resistance of 1 mi. of this wire is known to be 1.3 ohms; required the length of the wire.

5. If the resistance of 130 yds. of copper wire $\frac{1}{16}$ in. in diameter is 1 ohm, what is the resistance of 100 yds. of copper wire $\frac{1}{32}$ in. in diameter?

6. What is the resistance of 1 mi. of copper wire 1.14 mm in diameter, if the resistance of 1 mi. of copper wire 1.4 mm in diameter is 8.29 ohms?

7. What is the length of copper wire 1 mm in diameter which has the same resistance as 6 m of copper wire 0.74 mm in diameter?

8. What must be the length of an iron wire of sectional area 4 mm^2 to have the same resistance as a wire of pure copper 1000 m long whose sectional area is 1 mm^2, taking the conductivity of iron to be $\frac{1}{7}$ of that of copper?

9. How thick must an iron wire be in order that for the same length it shall offer the same resistance as a copper wire 2.5 mm in diameter? (Answer to 0.01 mm.)

10. If the resistance per cm^3 of a certain metal is $1.356 \cdot 10^{-5}$ ohm, what is the resistance of a wire of this metal 1 m long and 2 mm thick? (Answer to 10^{-5} ohm.)

11. What is the ratio of the resistances of two wires of the same metal, one of which is 30.48 cm long and weighs 35 g, while the other is 18.29 cm long and weighs 10.5 g?

12. The resistance of 1 m of pure copper wire 1 mm in diameter is 0.02 ohm, and the resistance of a certain specimen of copper wire 3 mm in diameter and 10 m long is 0.025; what is the ratio of their resistances?

13. The resistance of 1 mi. of a certain grade of copper wire whose diameter is 0.065 in. is 15.73 ohms, and the resistance of a wire of pure copper 1 ft. long and 0.001 in. in diameter is 9.94 ohms; what is the ratio of their conductivities?

14. The resistance of a certain dynamo is 10.9 ohms and the resistance of the rest of the circuit is 73 ohms; the electro-motive force of the machine being 839 volts, find how many amperes flow through the circuit.

$\because$ 1 volt forces 1 ampere through 1 ohm of resistance, 839 volts will force 839 amperes through 1 ohm of resistance, or $\dfrac{839 \text{ amperes}}{73 + 10.9}$ through $(73 + 10.9)$ ohms of resistance.

15. The resistance of a dynamo being 1.6 ohms and the resistance of the rest of the circuit being 25.4 ohms, and the electro-motive force being 206 volts; find how many amperes flow through the circuit.

16. Three arc lamps on a circuit have a resistance of 3.12 ohms each; the resistance of the wires is 1.1 ohms, and that of the dynamo is 2.8 ohms; find the voltage necessary to produce a current of 14.8 amperes.

17. Three arc lamps on a circuit have a resistance of 2.5 ohms each; the resistance of the wires is 0.5 ohm, and that of the dynamo is 0.5 ohm; find the voltage necessary to produce a current of 25 amperes through the circuit.

18. The resistance of a certain electric lamp is 3.8 ohms when a current of 10 amperes is flowing through it; what is the voltage?

19. If 31.1 volts force 35.8 amperes through a lamp, what is the resistance?

CHAPTER X.

Series.

A **series** is a succession of terms formed according to some common law.

E.g., in the following, each term is formed from the preceding as indicated:

$1, 3, 5, 7, \cdots\cdots$, by adding 2;
$7, 3, -1, -5, \cdots\cdots$, by subtracting 4, or by adding -4;
$3, 9, 27, 81, \cdots\cdots$, by multiplying by 3, or by dividing by $\frac{1}{3}$;
$64, 16, 4, 1, \frac{1}{4}, \cdots\cdots$, by dividing by 4, or by multiplying by $\frac{1}{4}$;
$2, 2, 2, 2, \cdots\cdots$, by adding 0, or by multiplying by 1.

In the series $0, 1, 1, 2, 3, 5, 8, 13, \cdots\cdots$, each term after the first two is found by adding the two preceding terms.

It is evident that the number of kinds of series is unlimited.

An **arithmetic series** (also called an arithmetic progression) is a series in which each term after the first is found by adding a constant to the preceding term.

E.g., $-7, -1, 5, 11, \cdots\cdots$, the constant being 6,
$2, 2, 2, 2, \cdots\cdots$, " " " 0,
$98, 66, 34, 2, \cdots\cdots$, " " " -32.

A **geometric series** (also called a geometric progression) is a series in which each term after the first is formed by multiplying the preceding term by a constant.

E.g., $2, 5, 12\frac{1}{2}, 31\frac{1}{4}, \cdots\cdots$, the constant being $2\frac{1}{2}$,
$3, -6, +12, -24, \cdots\cdots$, " " " -2,
$10, 5, 2\frac{1}{2}, 1\frac{1}{4}, \cdots\cdots$, " " " $\frac{1}{2}$,
$2, 2, 2, 2, \cdots\cdots$, " " " 1.

By custom, and because of their simplicity, arithmetic considers only arithmetic and geometric series. It should be stated, however, that the presence of this subject in applied arithmetic is merely a matter of tradition. It properly belongs to algebra, and hence may be omitted from the present course if Chap. XI is not taken. Since its application to business and to elementary science is slight, only a few of the more simple cases are considered.

I. Arithmetic Series.

Symbols. The following are in common use:

n, the number of terms of the series.
s, " sum " " " "
$t_1, t_2, t_3, \cdots\cdots t_n$, the terms of the series.
In particular, a, or t_1, the 1st term, and l, or t_n, the nth or last term.
d, the constant which added to any term gives the next; d is usually called the *difference*.

Formulae. There are two formulae in arithmetic series of such importance as to be designated as fundamental.

1. t_n, or $l = a + (n - 1)\,d$.

Proof. 1. $t_2 = a + d$ by definition.
$t_3 = t_2 + d = a + 2\,d$.
$t_4 = t_3 + d = a + 3\,d$.
$\vdots \quad \vdots \quad \vdots$
2. $\therefore t_n = t_{n-1} + d = a + (n - 1)\,d$.
3. Or $l = a + (n - 1)\,d$.

2. $s = \dfrac{n\,(a + l)}{2}$, or $\dfrac{n\,(t_1 + t_n)}{2}$.

Proof. 1. $s = a + (a + d) + (a + 2\,d) + \cdots\cdots (l - d) + l$.
2. Hence, $s = l + (l - d) + (l - 2\,d) + \cdots\cdots (a + d) + a$, by reversing the order.
3. $\therefore 2\,s = (a + l) + (a + l) + \cdots\cdots (a + l)$, by adding equations 1 and 2.
4. $\therefore 2\,s = n\,(a + l)$, $\because$ there is an $(a + l)$ in step 3 for each of the n terms in step 1.
5. $\therefore s = \dfrac{n\,(a + l)}{2}$.

It is evident that from formulae 1 and 2, various others can easily be obtained.

E.g., from $l = a + (n - 1)d$ it follows that

$$l - (n-1)d = a, \quad \frac{l-a}{n-1} = d, \text{ etc.}$$

From $s = \dfrac{n(a+l)}{2}$ it follows that

$$\frac{2s}{a+l} = n, \text{ etc.}$$

From $l = a + (n-1)d$ and $s = \dfrac{n(a+l)}{2}$ it follows that

$$s = \frac{n[a + a + (n-1)d]}{2}$$

$$= \frac{n[2a + (n-1)d]}{2}, \text{ etc.}$$

Exercises. 1. From $s = \dfrac{n(a+l)}{2}$, find a in terms of s, n, l.

2. From $l = a + (n-1)d$, find n in terms of l, a, d.
3. Find t_{10} in the series 540, 480, 420, ·····.
4. Find t_{100} in the series 1, 3, 5, ·····.
5. Find s, given $a = 1$, $l = 100$, $n = 100$.
6. Find s, given $a = 10$, $n = 6$, $d = -4$. Write out the series.
7. Find s, given $a = 40$, $n = 113$, $d = 5$.
8. Find n, given $s = 36{,}160$, $a = 40$, $l = 600$.
9. What is the sum of the first 50 odd numbers? the first 100? the first n?
10. What is the sum of the first 50 even numbers? the first 100? the first n?
11. What is the sum of the first 100 numbers divisible by 5? by 7?
12. $100 is placed at interest annually on the first of each January for 10 yrs., at 6%; find the total amount of principals and interest at the end of 10 yrs.
13. A body falling in vacuum would fall 4.9 m in the first second, and in each succeeding second it would fall 9.8 m farther than in the preceding second; how far would it fall in the 11th second? the 17th?
14. From the data of Ex. 13, how far would it fall in 11 secs.? 17 secs.? 57 secs.?
15. How long has a body been falling when it passes through 53.9 m during the last second?

II. Geometric Series.

Symbols. The following are in common use :

n, s, a, l, and t_1, t_2, ····· t_n, as in arithmetic series ;

r, the constant by which any term may be multiplied to produce the next ; r is usually called the *rate* or *ratio*.

Formulae. There are two formulae in geometric series of such importance as to be designated as fundamental.

1. t_n, or $l = ar^{n-1}$.

Proof. 1.
$$\begin{aligned} t_2 &= ar && \text{by definition.} \\ t_3 &= t_2 r &= ar^2. \\ t_4 &= t_3 r &= ar^3. \\ &\vdots \end{aligned}$$
2. $\therefore t_n = t_{n-1} r = ar^{n-1}$.
3. Or $l = ar^{n-1}$.

2. $s = \dfrac{ar^n - a}{r-1} = \dfrac{lr-a}{r-1}$.

Proof. 1. $s = a + ar + ar^2 + \cdots\cdots + ar^{n-2} + ar^{n-1}$.

2. $\therefore rs = \quad ar + ar^2 + \cdots\cdots + ar^{n-2} + ar^{n-1} + ar^n$, by multiplying by r.

3. $\therefore \ rs - s = ar^n - a$, by subtracting, (2) − (1).

4. $\therefore (r-1)s = ar^n - a$, and $s = \dfrac{ar^n - a}{r-1}$, by dividing by $(r-1)$.

5. And $\because ar^n = ar^{n-1} \cdot r = lr$, $\therefore s = \dfrac{lr-a}{r-1}$.

Exercises. 1. From $l = ar^{n-1}$, find a in terms of l, n, r.

2. Also r in terms of n, l, a.

3. From $s = \dfrac{ar^n - a}{r-1}$, find a in terms of r, n, s.

4. Find t_{11} in the series 1, 2, 4, 8, ·····; also in the series 1, $\frac{1}{2}$, $\frac{1}{4}$, ·····.

5. Find s, given $a = 2$, $r = 4$, $n = 3$; also, given $t_3 = 50$, $r = 5$, $n = 6$.

6. To what sum will \$1 amount, at 4% compound interest, in 5 yrs.? (Here $a = \$1$, $r = 1.04$, $n = 6$.)

7. To what sum will \$1 amount in 5 yrs., at 4% a year, compounded semi-annually ?

If the number of terms is infinite and $r < 1$, then s approaches as its limit $\frac{a}{1-r}$.

This is indicated by the symbols $s \doteq \frac{a}{1-r}$, n being infinite.

The symbol $\doteq$ is read "approaches as its limit."

Proof. 1. $\because r < 1$, the terms are becoming smaller, each being divided to obtain the next.

2. $\therefore l \doteq 0$, and $\therefore lr \doteq 0$, although they never reach that limit.

3. $\therefore s \doteq \frac{0-a}{r-1}$, by formula 2.

4. $\therefore s \doteq \frac{a}{1-r}$, by multiplying each term of the fraction by -1.

E.g., consider the series 1, $\frac{1}{2}$, $\frac{1}{4}$, ·····, where n is infinite. Here, $s \doteq \frac{a}{1-r}$, or $\frac{1}{1-\frac{1}{2}}$, or 2. That is, the greater the number of terms, the nearer the sum approaches 2, although it never reaches it for finite values of n.

Exercises. 1. Find the limits of the following sums, n being infinite:

(*a*) $10 + 5 + 2\frac{1}{2} + \cdots\cdots$

(*b*) $10 + (-5) + 2\frac{1}{2} + (-1\frac{1}{4}) + \cdots\cdots$

(*c*) $1 + 0.1 + 0.01 + \cdots\cdots$

(*d*) $90 + 9 + 0.9 + 0.09 + \cdots\cdots$

(*e*) $\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots\cdots$

(*f*) $\frac{7}{8} + \frac{7}{32} + \frac{7}{128} + \cdots\cdots$

2. Given $s \doteq 2\frac{1}{3}$, $r = \frac{1}{4}$, find a.

3. Given $s \doteq 4\frac{4}{9}$, $a = 4$, find r.

4. Given $s \doteq 1$, $r = \frac{299}{300}$, find a.

5. Find the sum of the first 20 terms of the following series: 32, 16, 8, 4, ·····.

6. Given $s = 155$, $r = 2$, $n = 5$, find a.

7. Given $s = 124.4$, $r = 3$, $n = 4$, find a.

8. Suppose an elastic ball falls 10 ft. and rebounds half that distance, then falls 5 ft. and rebounds half that distance, and so on, rebounding half the distance fallen each time; required the limit of the sum of the distances fallen; of the distances rebounded; of the total distances through which the ball passes.

9. If a man invests $1 at the beginning of each year at 4% compound interest, what will be the sum of the principals and interest at the end of 5 yrs.?

Circulating decimals. If the fraction $\frac{3}{11}$ is reduced to the decimal form, the result is $0.272727\cdots$, and similarly the fraction $\frac{11}{72} = 0.152777\cdots$. The former constantly repeats 27, and the latter constantly repeats 7 after 0.152.

When, beginning with a certain order of a decimal fraction, the figures constantly repeat in the same order, the number is called a *circulating, repeating,* or *recurring decimal,* and the part which repeats is called a *circulate, repetend,* or *recurring period.*

These various names are used, the subject being of too little practical importance to establish a uniform custom.

A circulate is usually represented by placing a dot over its first and last figures, thus:

$0.272727\cdots$ is represented by $0.\dot{2}\dot{7}$;
$0.152777\cdots$ " " " $0.152\dot{7}$.

A circulating decimal may be reduced to a common fraction by means of the formula $s \doteq \frac{a}{1-r}$, as in the following examples:

1. To what common fraction is $0.\dot{2}\dot{7}$ equal?

1. $0.\dot{2}\dot{7} = 0.27 + 0.0027 + 0.000027 + \cdots$
2. This is a geometric series with $a = 0.27$, $r = 0.01$, n infinite.
3. $\therefore s \doteq \frac{0.27}{1 - 0.01} = \frac{27}{99} = \frac{3}{11}$.

2. To what common fraction is $0.152\dot{7}$ equal?

1. $0.152\dot{7} = 0.152 + 0.0007 + 0.00007 + \cdots$
2. Of this, $0.0007 + 0.00007 + \cdots$ is a geometric series with $a = 0.0007$, $r = 0.1$, n infinite.
3. $\therefore s \doteq \frac{0.0007}{1-0.1} = \frac{7}{9000}$.
4. To this must be added 0.152, giving $0.152\frac{7}{9}$, or $\frac{1375}{9000}$, or $\frac{11}{72}$.

Exercise. Express as common fractions:

(a) $0.\dot{9}$.
(b) $0.\dot{0}\dot{1}$.
(c) $0.\dot{2}4\dot{7}$.
(d) $2.\dot{4}7\dot{6}$.
(e) $0.00\dot{3}71\dot{4}$.
(f) $0.12\dot{3}45\dot{0}$.

CHAPTER XI.

Logarithms.

ABOUT the year 1614 a Scotchman, John Napier, invented a system by which multiplication can be performed by addition, division by subtraction, involution by a single multiplication, and evolution by a single division.

This chapter might properly have followed the work on the four fundamental operations. By reserving it until this time, however, the practical application to scientific problems and the relation to series are more evident. It is not necessary for the understanding of the subsequent chapters and may, therefore, be omitted if desired. For the student who proposes to take even an elementary course in physics, however, the subject will be found of much value.

$2^0 = 1$	$2^6 = 64$
$2^1 = 2$	$2^7 = 128$
$2^2 = 4$	$2^8 = 256$
$2^3 = 8$	$2^9 = 512$
$2^4 = 16$	$2^{10} = 1024$
$2^5 = 32$	$2^{11} = 2048$

In considering the annexed series of numbers it is apparent that,

1. $\because 2^3 \cdot 2^5 = 2^8$,

$\therefore 8 \cdot 32 = 2^8 = 256$;

$\therefore$ the product can be found by adding the exponents $(3 + 5 = 8)$ and then finding what 2^8 equals.

2. $\because 2^9 : 2^3 = 2^6$,

$\therefore 512 : 8 = 64$; $\therefore$ this quotient can be found from the table by a single subtraction of exponents.

3. $\because (2^5)^2 = 2^5 \cdot 2^5 = 2^{10}$,

$\therefore 32^2 = 1024$.

4. $\because \sqrt{2^{10}} = \sqrt{2^5 \cdot 2^5} = 2^5$,

$\therefore \sqrt{1024} = 32$.

5. The exponents of 2 form an arithmetic series, while the powers form a geometric series.

In like manner a table of the powers of any number may be made and the four operations, multiplication, division, involution, evolution, reduced to the operations of addition, subtraction, multiplication, and division of exponents.

For practical purposes, *the exponents of the powers to which 10, the base of our system of counting, must be raised to produce various numbers* are put in a table, and these exponents are called the *logarithms* of those numbers.

In this connection the word "power" is used in its broadest sense, 10^n being considered as a power whether n is positive, negative, integral, or fractional. The logarithm of 100 is written "log 100."

E.g., $10^3 = 1000$, $\therefore \log 1000 = 3$. $10^0 = 1$, $\therefore \log 1 = 0$.
$10^2 = 100$, $\therefore \log 100 = 2$. $10^{-1} = \frac{1}{10}$, $\therefore \log 0.1 = -1$.
$10^1 = 10$, $\therefore \log 10 = 1$. $10^{-2} = \frac{1}{10^2}$, $\therefore \log 0.01 = -2$.

$10^{\frac{301}{1000}}$, that is, the thousandth root of 10^{301}, is nearly 2,

$$\therefore \log 2 = 0.301 \text{ nearly.}$$

Although log 2 cannot be expressed exactly as a decimal fraction, it can be found to any required degree of accuracy. In the present work logarithms are given to 4 decimal places; logarithms to 5 or 6 decimal places are sufficient for ordinary computations of considerable length.

Exercises. 1. What are the logarithms of these numbers? (*a*) 100,000; (*b*) $\frac{1}{1000}$, or 0.001; (*c*) 10^7; (*d*) 0.00001; (*e*) 10^{-6}; (*f*) $\sqrt{10}$, or $10^{\frac{1}{2}}$; (*g*) $\sqrt[7]{10}$.

2. What is meant by saying that the logarithm of 50 is 1.699? by saying that the logarithm of 300 is 2.4771?

3. Between what two consecutive integers does log 500 lie, and why? also log 2578? log 17? log 923,467?

4. Between what two consecutive negative integers does log 0.02 lie, and why? also log 0.007? log 0.0009? log 0.025?

5. What is the logarithm of $10^4 \cdot 10^6$? of $10^9 : 10^3$? of $\sqrt{10^8}$? of $(10^4)^5$?

6. What is the logarithm of $10^2 \cdot 10^3 \cdot 10^4$? of $10^4 \cdot 10^5 \cdot 10^7$? of 0.001 of $10^2 \cdot 10^3$?

7. What is the logarithm of $\sqrt{10^4 \cdot 10^6 \cdot 10^8}$? of $\sqrt[5]{10^{15} \cdot 10^{20}}$? of $\sqrt[10]{10}$?

Since 2473 lies between 1000 and 10,000, its logarithm lies between 3 and 4. It has been computed to be 3.3932. The integral part 3 is called the *characteristic* of the logarithm, and the fractional part 0.3932 the *mantissa*.

That is, $10^{\frac{33932}{10000}}$, or $10^{3.3932} = 2473$, $\therefore \log 2473 = 3.3932$.
$\because 10^{3.3932} : 10^1 = 10^{2.3932}$, $\therefore 10^{2.3932} = 247.3$, $\therefore \log 247.3 = 2.3932$.
Similarly, $10^{1.3932} = 24.73$, $\therefore \log 24.73 = 1.3932$.
" $10^{0.3932} = 2.473$, $\therefore \log 2.473 = 0.3932$.
" $10^{0.3932-1} = 0.2473$, $\therefore \log 0.2473 = 0.3932-1$.

It is thus seen that

1. The characteristic can always be found by inspection.

Thus, because 438 lies between 100 and 1000, hence log 438 lies between 2 and 3, and log 438 = 2 + some mantissa.

Similarly, 0.0073 lies between 0.001 and 0.01, hence log 0.0073 lies between − 3 and − 2, and log 0.0073 = − 3 + some mantissa.

2. The mantissa is the same for any given succession of digits, wherever the decimal point may be.

Thus, log 2473 = 3.3932, and log 0.2473 = 0.3932 − 1.

3. Therefore, only the mantissas need be put in a table.

Instead of writing the negative characteristic after the mantissa, it is often written before it, but with a minus sign above; thus, log $0.2473 = 0.3932 - 1 = \bar{1}.3932$, this meaning that only the characteristic is negative, the mantissa remaining positive.

Negative numbers are not considered as having logarithms, but operations involving negative numbers are easily performed. *E.g.*, the multiplication expressed by $1.478 \cdot (-0.007283)$ is performed as if the numbers were positive, and the proper sign is prefixed.

Exercises. 1. What is the characteristic of the logarithm of a number of 3 integral places? of 5? of 10? of n?

2. What is the characteristic of the logarithm of 0.2? of any decimal fraction whose first significant figure is in the first decimal place? the second decimal place? the 10th? the nth?

3. From Exs. 1, 2 formulate a rule for determining the characteristic of the logarithm of any positive number.

4. If log 39,703 = 4.5988, what are the logarithms of (*a*) 39,703,000, (*b*) 397.03, (*c*) 3.9703, (*d*) 0.00039703?

The fundamental theorems of logarithms.

I. *The logarithm of the product of two numbers equals the sum of their logarithms.*

1. Let $a = 10^m$, then $\log a = m$.
2. Let $b = 10^n$, " $\log b = n$.
3. $\therefore ab = 10^{m+n}$, and $\log ab = m + n = \log a + \log b$.

II. *The logarithm of the quotient of two numbers equals the logarithm of the dividend minus the logarithm of the divisor.*

1. Let $a = 10^m$, then $\log a = m$.
2. Let $b = 10^n$, " $\log b = n$.
3. $\therefore \frac{a}{b} = \frac{10^m}{10^n} = 10^{m-n}$, and $\log \frac{a}{b} = m - n$.

III. *The logarithm of the* n*th power of a number equals* n *times the logarithm of the number.*

1. Let $a = 10^m$, then $\log a = m$.
2. $\therefore a^n = 10^{mn}$, and $\log a^n = nm = n \log a$.

IV. *The logarithm of the* n*th root of a number equals* $\frac{1}{n}$*th of the logarithm of the number.*

1. Let $a = 10^m$, then $\log a = m$.
2. $\therefore a^{\frac{1}{n}} = 10^{\frac{m}{n}}$, and $\log a^{\frac{1}{n}} = \frac{m}{n} = \frac{1}{n} \cdot \log a$.

Th. III might have been stated more generally, so as to include Th. IV, thus: $\text{Log } a^{\frac{x}{y}} = \frac{x}{y} \cdot \log a$. The proof would be substantially the same as in Ths. III and IV.

Exercises. Given $\log 2 = 0.3010$, $\log 3 = 0.4771$, $\log 5 = 0.6990$, $\log 7 = 0.8451$, and $\log 514 = 2.7110$, find the following:

1. $\log 6$.
2. $\log 14$.
3. $\log 7^{10}$.
4. $\log \sqrt{2}$.
5. $\log 42$.
6. $\log 5^{\frac{1}{4}}$.
7. $\log 105$.
8. $\log 1.05$.
9. $\log \sqrt{514}$.
10. $\log 514^2$.
11. $\log 1542$.
12. $\log 257$.
13. $\log 1799 [= \log(\frac{1}{2} \cdot 514 \cdot 7)]$.
14. $\log \sqrt[7]{3^4}$.
15. $\log \sqrt{21}$.
16. Show how to find $\log 5$, given $\log 2$.

N	0	1	2	3	4	5	6	7	8	9
0	0000	0000	3010	4771	6021	6990	7782	8451	9031	9542
1	0000	0414	0792	1139	1461	1761	2041	2304	2553	2788
2	3010	3222	3424	3617	3802	3979	4150	4314	4472	4624
3	4771	4914	5051	5185	5315	5441	5563	5682	5798	5911
4	6021	6128	6232	6335	6435	6532	6628	6721	6812	6902
5	6990	7076	7160	7243	7324	7404	7482	7559	7634	7709
6	7782	7853	7924	7993	8062	8129	8195	8261	8325	8388
7	8451	8513	8573	8633	8692	8751	8808	8865	8921	8976
8	9031	9085	9138	9191	9243	9294	9345	9395	9445	9494
9	9542	9590	9638	9685	9731	9777	9823	9868	9912	9956
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
N	0	1	2	3	4	5	6	7	8	9

N	0	1	2	3	4	5	6	7	8	9
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
N	0	1	2	3	4	5	6	7	8	9

Explanation of table. *Given a number to find its logarithm.* In the table on pp. 114 and 115 only the mantissas are given. For example, in the row opposite 71, and under 0, 1, 2, ····· will be found:

N	0	1	2	3	4	5	6	7	8	9
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567

This means that the mantissa of log 710 is 0.8513, of log 711 it is 0.8519, and so on to log 719. Hence,

$$\log 715 = 2.8543, \qquad \log 7.18 = 0.8561,$$
$$\log 71{,}600 = 4.8549, \qquad \log 0.0719 = \bar{2}.8567.$$

And $\because$ 7154 is $\frac{4}{10}$ of the way from 7150 to 7160, $\therefore$ log 7154 is about $\frac{4}{10}$ of the way from log 7150 to log 7160.

$\therefore$ log 7154 = log 7150 + $\frac{4}{10}$ of the difference between log 7150 and log 7160

$$= 3.8543 + \tfrac{4}{10} \text{ of } 0.0006$$
$$= 3.8543 + 0.0002 = 3.8545.$$

The above process of finding the logarithm of a number of four significant figures is called *interpolation*. It is merely an approximation available within small limits, since numbers do not vary as their logarithms, the numbers forming a geometric series while the logarithms form an arithmetic series. It should be mentioned again that the mantissas given in the table are only approximate, being correct to 0.0001. This is far enough to give a result which is correct to three figures in general, and usually to four, an approximation sufficiently exact for many practical computations.

In all work with logarithms, the characteristic should be written before the table is consulted, even if it is 0. Otherwise it is liable to be forgotten, in which case the computation will be valueless.

Exercises. From the table find the following:

1. log 38.
2. log 743.
3. log 14,000.
4. log 3940.
5. log 3.81.
6. log 0.00123.
7. log 1855.
8. log 23.41.
9. log 1.823.
10. log 0.2769.
11. log 0.00001727.
12. $\log \sqrt{4.28}$.
13. $\log 9.821^3$.
14. $\log 75.55^{\frac{1}{5}}$.
15. $\log 0.0129^5$.
16. $\log \sqrt[3]{125}$.

Given a logarithm to find the corresponding number. The number to which a logarithm corresponds is called its *antilogarithm.*

$$E.g., \because \log 2 = 0.3010, \quad \therefore \text{antilog } 0.3010 = 2.$$

The method of finding antilogarithms will be seen from a few illustrations. Referring again to the row after 71 on p. 115, we have:

N	0	1	2	3	4	5	6	7	8	9
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567

Hence, we see that

antilog $0.8513 = 7.1$, antilog $5.8531 = 713{,}000$,

antilog $\bar{2}.8567 = 0.0719$, antilog $\bar{1}.8555 = 0.717$.

Furthermore, $\because$ 8540 is $\frac{1}{2}$ way from 8537 to 8543,

$\therefore$ antilog 2.8540 is about $\frac{1}{2}$ way from antilog 2.8537 to antilog 2.8543.

$\therefore$ antilog 2.8540 is about $\frac{1}{2}$ way from 714 to 715.

$\therefore$ antilog $2.8540 = 714.5$.

Similarly, to find antilog $\bar{1}.8563$.

$$\begin{array}{ll} \text{antilog } \bar{1}.8567 = 0.719 & \qquad \bar{1}.8563 \\ \text{antilog } \underline{\bar{1}.8561} = 0.718 & \qquad \underline{\bar{1}.8561} \\ \phantom{\text{antilog } \bar{1}.856}6 & \qquad \phantom{\bar{1}.856}2 \end{array}$$

$$\therefore \text{antilog } \bar{1}.8563 = 0.718\tfrac{2}{6} = 0.7183.$$

The interpolation here explained is, as stated on p. 116, merely a close approximation; it cannot be depended upon to give a result beyond four significant figures except when larger tables are employed.

Exercises. From the table find the following:

1. antilog 0.1234.
2. antilog 3.4271.
3. antilog $\bar{2}.8193$.
4. antilog 4.2183.
5. antilog 1.9286.
6. antilog $\bar{1}.7829$.
7. antilog $0.9485 - 3$.
8. antilog 5.7834.
9. antilog 0.9996.
10. antilog $0.6585 - 5$.
11. antilog 10.5441.
12. antilog 2.0000.
13. antilog $0.6120 - 1$.
14. antilog 0.7070.
15. antilog 1.7850.
16. antilog $0.9290 - 2$.
17. antilog 3.8320.
18. antilog 3.6387.

Cologarithms. In cases of division by a number n it is often more convenient to add the logarithm of $\frac{1}{n}$ than to subtract the logarithm of n. The logarithm of $\frac{1}{n}$ is called the *cologarithm* of n.

$$\because \log \frac{1}{n} = \log 1 - \log n = 0 - \log n,$$

$$\therefore \operatorname{colog} n = - \log n.$$

Also, $\operatorname{colog} n = 10 - \log n - 10$, often a more convenient form to use.

The cologarithm can evidently be found by subtracting each digit from 9, excepting the right-hand significant one (which must be subtracted from 10) and the zeros following, and then subtracting 10.

E.g., to find colog 6178.

$$\begin{array}{rl} & 9.\ 9\ \ 9\ \ 9\ 10 \\ \log 6178 = & \underline{3.\ 7\ \ 9\ \ 0\ \ 9} \\ \operatorname{colog} 6178 = & 6.\ 2\ \ 0\ \ 9\ \ 1 - 10. \end{array}$$

To find colog 41.5.

$$\begin{array}{rl} & 9.\ 9\ \ 9\ 10\ \ 0 \\ \log 41.5 = & \underline{1.\ 6\ \ 1\ \ 8\ \ 0} \\ \operatorname{colog} 41.5 = & 8.\ 3\ \ 8\ \ 2\ \ 0 - 10. \end{array}$$

To find colog 0.013.

$$\begin{array}{rl} & 9.\ 9\ \ 9\ \ 9\ 10 \\ \log 0.013 = & \underline{\bar{2}.\ 1\ \ 1\ \ 3\ \ 9} \\ \operatorname{colog} 0.013 = & 11.\ 8\ \ 8\ \ 6\ \ 1 - 10 = 1.8861. \end{array}$$

In case the characteristic exceeds 10 but is less than 20, colog n may be written $20 - \log n - 20$, and so for other cases; but these cases are so rare that they may be neglected at this time.

The advantage of using cologarithms will be apparent from a single example:

To find the value of $\frac{317 \cdot 92}{6178 \cdot 0.13}$.

Using Cologarithms.	Not using Cologarithms.
$\log 317 = 2.5011$	$\log 317 = 2.5011$
$\log 92 = 1.9638$	$\log 92 = \underline{1.9638}$
$\text{colog}\ 6178 = 6.2091 - 10$	$\underline{\log (317 \cdot 92) = 4.4649}$
$\text{colog}\ 0.13 = \underline{10.8861 - 10}$	$\log 6178 = 3.7909$
$\log 36.32 = 1.5601$	$\log 0.13 = \underline{\bar{1}.1139}$
	$\underline{\log (6178 \cdot 0.13) = 2.9048}$
	$\log (317 \cdot 92) = 4.4649$
	$\log (6178 \cdot 0.13) = \underline{2.9048}$
	$\log 36.32 = 1.5601$

Computations by logarithms. A few illustrative problems will now be given covering the types which the student will most frequently meet. It is urged that all work be neatly arranged, since as many errors arise from failure in this respect as from any other single cause.

Since π enters so frequently into computations, the following logarithms will be found useful:

$$\log \pi = 0.4971, \qquad \log \frac{1}{\pi} = \bar{1}.5029.$$

1. Find the value of $\dfrac{0.007^3}{0.03625}$.

$$\begin{aligned}
\log 0.007 &= \underline{0.8451 - 3} \\
3 \cdot \log 0.007 &= 2.5353 - 9 \\
\text{colog}\ 0.03625 &= \underline{11.4407 - 10} \\
&\ 13.9760 - 19 \\
&= 0.9760 - 6 = \log 0.000009462.
\end{aligned}$$

$$\therefore 9.462 \cdot 10^{-6} = Ans.$$

It will be noticed that the negative characteristic is less confusing if written by itself at the right.

2. Find the value of $0.09515^{\frac{1}{3}}$.

$$\log 0.09515 = 0.9784 - 2.$$

$\because$ the characteristic (-2) is not divisible by 3, this may be written

$$\log 0.09515 = 1.9784 - 3.$$

Then $$\tfrac{1}{3} \log 0.09515 = 0.6595 - 1 = \log 0.4566.$$

$$\therefore 0.4566 = Ans.$$

3. Find the value of $\dfrac{2.706 \cdot 0.3 \cdot 0.001279}{86090}$.

This may at once be written $\dfrac{2.706 \cdot 3 \cdot 1.279}{8.609} \cdot 10^{-8}$, thus simplifying the characteristics. Then

$$\begin{aligned}
\log 2.706 &= 0.4324\\
\log 3 &= 0.4771\\
\log 1.279 &= 0.1069\\
\operatorname{colog} 8.609 &= \underline{9.0650 - 10}\\
\log 1.206 &= 0.0814\\
\therefore 1.206 \cdot 10^{-8} &= Ans.
\end{aligned}$$

4. Given $2^x = 7$, find x, the result to be correct to 0.01.

$$x \log 2 = \log 7.$$

$$\therefore x = \frac{\log 7}{\log 2} = \frac{0.8451}{0.3010} = 2.81.$$

This division may be performed directly or by finding the antilogarithm of ($\log 0.8451 - \log 0.3010$), the former being the more expeditious in this case.

5. The weight of an iron sphere, specific gravity 7.8, is 14.3 kg; find the radius.

$$v = \tfrac{4}{3}\pi r^3 \cdot 1 \text{ cm}^3 = \text{volume in cm}^3.$$

$$\therefore \text{weight} = \tfrac{4}{3}\pi r^3 \cdot 7.8 \cdot 1 \text{ g} = 14{,}300 \text{ g}.$$

$$\therefore r = \left(\frac{3 \cdot 14300}{4\pi \cdot 7.8}\right)^{\frac{1}{3}}, \text{ the number of centimeters of radius.}$$

$$\begin{aligned}
\log 3 &= 0.4771\\
\log 14{,}300 &= 4.1553\\
\operatorname{colog} 4 &= 9.3979 - 10\\
\operatorname{colog} \pi &= 9.5029 - 10\\
\operatorname{colog} 7.8 &= \underline{9.1079 - 10}\\
3&\,\big|\,\underline{2.6411}\\
\log 7.593 &= 0.8804 \qquad \therefore \text{radius} = 7.593 \text{ cm.}
\end{aligned}$$

Exercises. 1. Find the value of $(12.8 \div 0.07235)^{\frac{1}{2}}$.

2. Find the value of $(42 \cdot 9.37)^{\frac{1}{3}} \div 0.127^{\frac{1}{2}}$.
3. Find the value of $(4.376/\pi)^{\frac{1}{3}}$.
4. Given $x : 4.127 = 0.125 : 2736$; find x.
5. Given $x^3 = x^5 : 5$; find x.
6. Find the value of $27^{\frac{1}{27}}$.
7. Given $117{,}600 = 7^{n-1}$; find n, correct to 0.1.
8. Find the value of $\sqrt{\pi \cdot 4.927}$.

9. Given $0.47 : x = x : 1.249$; find x.

10. Find the value of $0.00234 \cdot 72.28 \cdot 5.126 \cdot 10^{-7}$.

11. What power will just raise a weight of 17.5 lbs., the fulcrum of the lever being 1.73 ft. from the weight and 4.19 ft. from the power ?

12. At what distance from the fulcrum must a power of 91 lbs. be exerted to raise a weight of 7493 lbs. 2 ft. 3 in. from the fulcrum ?

13. It is shown in studying the strength of materials that a cylindrical iron shaft $5\frac{3}{4}$ in. in diameter and 5 ft. 7.5 in. long between its supports will support a load at the center of $0.726 \cdot d \cdot 8700$ lbs. $/l$, where $d =$ the number of inches of diameter and $l =$ the number of feet of length. Perform the computation.

14. 2240 lbs. of chalk occupy 15.5 cu. ft., and a cubic foot of water weighs $62\frac{1}{2}$ lbs. ; what is the specific gravity of chalk ?

15. The surface of a sphere is 4 sq. in.; what is its volume ?

16. The volume of a sphere is 1 cu. ft.; what is its radius ?

17. What is the specific gravity of a substance of which a sphere of radius 9 cm weighs 15 kg ?

18. What is the weight of a silver cone of radius 2 cm and height 3.6 cm, the specific gravity of silver being 10.5 ?

19. If the intensity of light varies inversely as the square of the distance and directly as the illuminating power of its source, what is the ratio of the intensities of a candle 3.75 ft. distant and a 41.5 candle lamp 13.2 ft. distant ? (Answer to 0.01.)

20. If the intensity of the light of the full moon is found to be equal to that of a candle at a distance of 4 ft., what is the equivalent in candle power of the moon ? (See Ex. 19 ; take the distance of the moon as $2.4 \cdot 10^5$ mi.)

[Omit the following if the problems in electricity were not taken.]

21. The resistance of 1 cm^3 of copper is $1.6 \cdot 10^{-6}$ ohm at 0° C. and the resistance increases by $3.9 \cdot 10^{-3}$ of this for each degree rise in temperature ; find the resistance of a wire 10 m long and 1 mm in diameter at 25° C.

22. An incandescent lamp takes a current of 0.71 ampere and the electro-motive force is 98.5 volts; what is the resistance of three such lamps ?

23. If an incandescent lamp of $81\frac{1}{3}$ ohms resistance takes a current of 0.756 ampere, what is the voltage ?

24. A mile of telegraph wire 2 mm in diameter offers a resistance of 12.85 ohms ; what is the resistance of 439 yds. of wire of the same material 0.8 mm in diameter ?

CHAPTER XII.

Graphic Arithmetic.

WHEN the mind seeks to clearly appreciate the relations between several measurements, it is of great value to resort to a graphic representation, accurately drawn to a scale. Thus, to say that the distance in millions of miles from the Sun to Mercury is 36, to Earth 92, to Jupiter 481, and to Neptune 2778 is not nearly as expressive as when accompanied by the following graphic representation of these measurements:

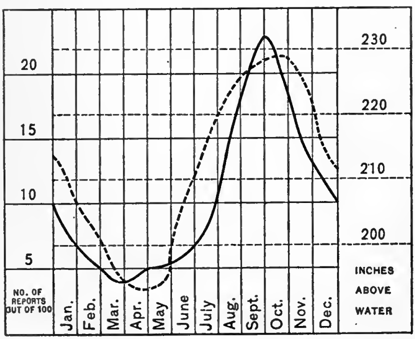


FIG. 21.

The annexed curve has been plotted to represent graphically the statistics compiled by a state board of health with reference to typhoid fever and the condition of wells for the various months in the year.

The dotted line shows the average number of inches above the surface of water in the wells,

and hence is highest when the water is lowest; the black line shows the number of reports, out of every 100, in which typhoid fever was mentioned as prevalent. The effect of low water upon typhoid fever is thus much more clearly seen than it would be from the mere numbers.

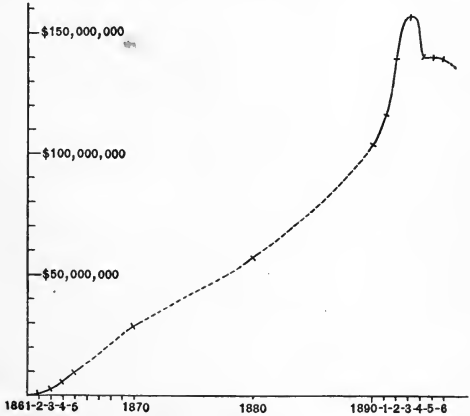


FIG. 22.

The above curve represents the sums paid by the United States government for pensions in various years as follows (by millions of dollars):

1862, 0.8; 1863, 1; 1864, 5; 1865, 8; 1870, 28; 1880, 57; 1890, 106; 1891, 119; 1892, 141; 1893, 158; 1894, 141; 1895, 141; 1896, 139.

Exercises. 1. In cases of contagious diseases the premises should be isolated and disinfected. In a certain part of the country, where this was neglected, the number of cases and deaths to each outbreak of diphtheria averaged respectively 13.78 and 3.81; in the same state, but where these precautions were taken, the corresponding results were 2.45 and 0.69. Represent graphically by four straight lines.

2. Represent graphically the following per capita indebtedness as given in recent government reports: Austria-Hungary $71, France $116, Prussia $37, Great Britain and Ireland $88, Italy $76, Russia $31, Spain $74, United States $15.

3. A *foot-ton* is the measure of force required to raise 1 ton 1 ft. Plot the five curves corresponding to the following statistics and show where the growth has been the most marked.

Year.	Millions of Foot-tons Daily.				Foot-tons Daily per Inhabitant.
	Hand.	Horse.	Steam.	Total.	
1820	753	3,300	240	4,293	446
1840	1,406	12,900	3,040	17,346	1,020
1860	2,805	22,200	14,000	39,005	1,240
1880	4,450	36,600	36,340	77,390	1,545
1885	6,406	55,200	67,700	129,306	1,940

4. Represent the following statistics graphically:

	Millions of Foot-tons Daily.				Foot-tons per Inhabitant.
	Hand.	Horse.	Steam.	Total.	
United States	6,406	55,200	67,700	129,306	1,940
Great Britain	3,210	6,100	46,800	56,110	1,470
Germany	4,280	11,500	29,800	45,580	902
France	3,380	9,600	21,600	34,580	910
Austria	3,410	9,900	9,200	22,510	560
Italy	2,570	4,020	4,800	11,390	380
Spain	1,540	5,500	3,600	10,640	590

5. Of those persons in England and Wales marrying in 1843, 327 out of every 1000 of the men could not write and 490 out of every 1000 of the women; the numbers for each succeeding tenth year to 1893 were as follows: 304 and 439, 238 and 331, 188 and 254, 126 and 155, 50 and 57; represent these statistics by two curves.

6. The average annual mortality from smallpox in Sweden has been as follows:

1774–1801, before vaccination,	2045
1802–1816, vaccination allowed,	480
1817 to the present, vaccination compulsory,	155

Represent graphically.

7. Represent graphically the annexed statistics concerning the population of the United States.

Population of the United States.		
Decades.	Total Whites.	Foreign Whites.
1750	1,040,000	
1760	1,385,000	
1770	1,850,000	
1780	2,383,000	
1790	3,177,257	
1800	4,306,446	44,282
1810	5,862,073	96,725
1820	7,862,166	176,825
1830	10,537,378	315,830
1840	14,195,805	859,202
1850	19,553,068	2,244,602
1860	26,922,537	4,138,697
1870	33,589,377	5,507,229
1880	43,402,970	6,679,943
1890	54,983,890	9,249,547

8. The population of the world is estimated at 1480 millions distributed as follows (in millions): Europe 357.4, Africa 164, Asia 826, Australia 3.2, the Americas 121.9, Oceanica and the Polar regions 7.5. Represent graphically, first arranging in order of magnitude.

9. The distance from the sun to the earth is about $93 \cdot 10^6$ mi., to Neptune 30 times as far, to the nearest fixed star $256 \cdot 10^{11}$; represent these graphically by measurements on a single straight line.

10. In a certain city, before the strict enforcement of the law requiring milk and cream to be of a certain grade, the following are the per cents of samples found below grade for the various weeks from May 1 to Aug. 1 inclusive: 50, 64, 42, 45, 42, 35, 35, 38, 37, 39, 45, 37, 28, 50; for the corresponding weeks of the next year, when the law was strictly enforced, the per cents are 4, 3, 2, 2, 6, 4, 7, 7, 7, 5, 10, 5, 7, 5. Represent the two by broken lines on the same diagram.

11. The indebtedness of the United States government at the various periods named was as follows (in tens of millions of dollars): 1791, 7.5; 1800, 8.3; 1810, 5.3; 1816, 12.7; 1820, 9.1; 1830, 4.9; 1835, 0.0004; 1840, 0.5; 1850, 6.3; 1860, 6.4; 1862, 52.4; 1863, 112; 1865, 268; 1866, 277; 1867, 268; 1868, 261; 1870, 248; 1880, 213; 1890, 155; 1896, 179. Represent these statistics graphically, noting that the differences in dates are not uniform.

12. The tonnage of the merchant ships of America and England for the various years is here stated in millions; represent the statistics by two curves on the same diagram. America: 1850, 3; 1860, 5; 1870, 4; 1880, 4; 1890, 4.4; 1892, 4.8; 1894, 4.7; 1896, 4.7. England: 1850, 4; 1860, 6; 1870, 7; 1880, 8; 1890, 11.6; 1892, 12.5; 1894, 13.2; 1896, 13.6.

CHAPTER XIII.

Introduction to Percentage.

WITH the introduction of the metric system throughout a large part of the world and the almost universal use of the decimal system of money save in Great Britain and some of her dependencies, the subject of decimal fractions has in modern times become one of great importance. It has come to be common to reckon by tenths, hundredths, and thousandths, and the subject of computation by hundredths has received the special name *percentage*. The subject requires no principles differing from those used in operating with common and decimal fractions, and the problems require no methods for solution other than those already discussed. There is, therefore, no reason for not treating percentage with decimal fractions, as was done to some extent, except that it is especially needed in the business arithmetic which is now to be considered.

Common terms. *Per cent* means *hundredths* (*hundredth, of a hundredth*), the words, as used in America, always being interchangeable within grammatical limits. The symbol % means, therefore, either *per cent* or *hundredths* (*hundredth, of a hundredth*).

$\frac{6}{100}$, 6%, 0.06 are each read 6 per cent, or 6 hundredths.
$\frac{6}{100}$%, 0.06%, 0.0006 " " 6 hundredths per cent, 6 hundredths of a hundredth, 6 hundredths of 1 per cent, or 6 ten-thousandths.

0.01, 1%, $\frac{1}{100}$ are each read 1 per cent or 1 hundredth.
0.00½, ½% " " " ½ of a hundredth, ½ per cent, or ½ of 1 per cent, and each equals $\frac{1}{200}$.

The words "per cent" are sometimes taken to mean "out of 100," 6% then meaning "6 out of 100." 200% would not, however, be so clearly understood by this explanation.

If a certain per cent (meaning a certain number of hundredths) of a number is to be taken, as 6%, the 6% is called a *rate*, the 6 being called the *rate per cent.*

Thus, if the *rate of interest* in a certain bank is 4%, the *rate per cent of interest* is 4.

Illustrative Exercises.

I. 1.55 is what per cent of 15½?

1. Let $r\% =$ the rate.
2. Then $r\%$ of $15\frac{1}{2} = 1.55$.
3. $\therefore r\% = \dfrac{1.55}{15\frac{1}{2}} = .10$, by dividing equals by 15½. Ax. 7

II. 69.35 is 9½% of what number?

1. Let $n =$ the number.
2. Then $9\frac{1}{2}\%$ of $n = 69.35$.
3. $\therefore n = \dfrac{69.35}{0.09\frac{1}{2}} = 730$, by dividing equals by 9½%. Ax. 7

III. What is 10% of $634?

10% of \$634 = \$63.40. The question is simply, "What is 0.1 of \$634?" No more analysis should be required than in the problem, "Find what 2 × \$3 equals."

IV. After deducting 9½% of a number there remains 660.65; required the number.

1. Let $n =$ the number.
2. Then $n - 9\frac{1}{2}\% \, n$, or $90\frac{1}{2}\% \, n$, equals 660.65.
3. $\therefore n = \dfrac{660.65}{0.90\frac{1}{2}} = 730$, by dividing equals by 90½%. Ax. 7

V. After adding 9½% of a number to that number the sum is 799.35; required the number.

1. Let $n =$ the number.
2. Then $n + 9\frac{1}{2}\% \, n$, or $109\frac{1}{2}\% \, n$, equals 799.35.
3. $\therefore n = \dfrac{799.35}{1.09\frac{1}{2}} = 730$, by dividing equals by 109½%. Ax. 7

Exercises. 1. The United States silver dollar weighs 26.729 g and the Japanese silver dollar (or yen) weighs 26.9564 g; each is 0.900 fine (*i.e.*, 90% pure silver); how many grams of fine silver (*i.e.*, pure silver) in each? How do you check your result?

2. An English sovereign weighs 7.9881 g and is 0.916 fine; how many grams of fine gold does it contain?

3. The British nautical unit of length is the knot, 6080 ft.; the common mile is what per cent of the knot?

4. A fathom being strictly 0.1% of a knot, this is what per cent of the 6-ft. fathom?

5. Of 1486 graduates of women's colleges in England, recently questioned, 680 were teachers, 208 were married, 13 were physicians or nurses, and the rest were in various professions or trades. What per cent of the graduates were teachers? married? physicians or nurses? in other work? How do you check your result?

6. The following table shows the values of the total exported merchandise of the United States for the several years, and of the manufactured part. Find what per cent the manufactured part is of the total in each year.

YEAR.	TOTAL.	MANUFACTURED.
1860	$316,242,423	$40,345,892
1870	455,208,341	68,279,764
1880	823,946,353	102,856,015
1890	845,293,828	151,102,376
1895	793,397,890	183,595,743

7. From the data of Ex. 6 find the rate of increase of each amount over that of the preceding period.

8. The ratio of the arid and semi-arid regions of the United States (excluding Alaska) to the rest of the country is about 24 : 25; at this ratio, what per cent of our territory is arid and what per cent is semi-arid?

9. From the following table showing the wealth of the United States and the average wealth of each inhabitant, compute the rate of increase in each from period to period.

CENSUS.	MILLIONS OF DOLLARS.	DOLLARS PER INHABITANT.
1820	1,960	205
1840	3,910	230
1860	16,160	514
1880	43,642	870
1890	65,037	1,039

10. Cinnabar consists of two substances, sulphur and mercury, in the ratio of 7 parts (by weight) of the former to 44 of the latter. The weight of the sulphur is what per cent of the weight of the mercury? That of the mercury is what per cent of that of the sulphur? The weight of each is what per cent of the weight of the cinnabar? How many grams of each in 178.5 g of cinnabar?

11. A dealer is obliged to sell sugar so that for 43.5 lbs. he receives as much as 36 lbs. cost; did he gain or lose, and what rate per cent?

12. At the time of a recent census in Ireland 38,121 people, or 0.81% of the total population, could speak the Irish language only; required the population at that time, correct to 1000.

13. In 1894 the population of London was 4,349,116, an increase of about 3.28% over the population in 1891; required the population in 1891, correct to 1000.

14. National banks are required to keep on hand 25% of their deposits; find if these banks have complied with the law and give the per cent in each case:

	Specie on Hand.	Other Legal Tender.	Deposits.
(*a*) Bank of N. Y.	$2,050,000	$1,200,000	$12,170,000
(*b*) Manhattan Bank	2,608,000	3,219,000	16,154,000
(*c*) Nassau Bank	192,000	539,400	2,885,000
(*d*) German Exchange Bank	267,900	587,400	3,140,500
(*e*) Germania Bank	511,400	541,400	4,196,300

15. A certain bank has on hand $294,600 in specie and $325,400 in other legal tender, and this sum is 26.4% of the deposits; find, correct to $1000, the amount of the deposits.

16. A report made by the banks of New York City shows $164,172,200 in cash on hand, this being 31.3% of the total deposits; find, correct to $1000, the amount of the deposits.

17. In one year the imports of specie into New York amounted to $83,233,962, and the exports to $102,487,994; the difference was what per cent of the imports? of the exports?

18. An insurance company charges a premium of $22.50 for insuring a house for $1500 for 3 yrs.; what is the rate for the 3 yrs.?

19. A book agent sells during the summer 300 books at $2.75 each; he is allowed 40% of the receipts; how much does he earn?

20. A man invests $3000 in property which he rents for $228 a year. The taxes are $33, insurance is $18, water tax $5, repairs $47; what per cent does he receive on his investment?

21. What is the per cent of attendance in a schoolroom of 59 pupils when there are $29\frac{1}{2}$ da. of absence in 4 school weeks? when there are 10 da. of absence in 1 school week?

22. A man sold two horses for $125 each; on the purchase price of one he gained 20% and on that of the other he lost 20%; what was his total gain or loss?

23. The distances between the following cities by the present routes of sea travel and also by the proposed Nicaragua Canal are given below; required the gain per cent by the canal over the present route, in each case.

	Between	Miles, Present Route, via		Miles via Nicaragua Canal.
(a)	New York and San Francisco	Cape Horn	15,660	4,907
(b)	New York and Puget Sound	Magellan	13,935	5,665
(c)	New York and Hong Kong	Cape G. H.	13,750	10,695
(d)	New York and Melbourne	Cape Horn	13,760	9,882
(e)	Liverpool and San Francisco	Cape Horn	15,620	7,627
(f)	New Orleans and San Francisco	Cape Horn	16,000	4,147

24. A man has the following investments: $2000 which yields 4%, $450 which yields 6%, and $1200 which yields $6\frac{1}{2}$%. He can invest the whole amount so that it will yield 5%; would he gain or lose by so doing, and what per cent of the whole sum?

25. What is the premium for insuring a house for $3000 for 3 yrs. at $1\frac{1}{2}$% for the whole time? at $2\frac{1}{4}$%? at 75 cts. per $100? at $7 per $1000?

26. It is estimated that about 600,000,000 passengers are carried on steamboats in the United States in one year and that about 70 are killed; what per cent are not killed?

27. About 590,000,000 passengers are carried on railways in the United States in one year and about 300 are killed; what per cent are not killed?

28. About $75\frac{3}{5}$% of the number of patents granted by the United States in the 60 yrs. beginning with 1837 represents the number of patents refused; the total number of applications was 993,953; how many patents were granted and how many refused? (Answer correct to 1000.)

29. By the eleventh census the number of Indians on reservations under control of the Indian Office was 106% as large as the rest of the Indians; the total number was 249,000; how many were in each of these two groups? (Answer correct to 1000.)

30. About 74% of the territory of the United States, excluding Alaska and the Indian reservations, is cleared land, and 495,000,000 acres are forest; what is the total area, excluding the portions specified?

31. What per cent of the 343,267 immigrants landing in the United States in a certain year did not rank among either the 46,807 skilled laborers or the 2324 professional men?

32. The chief export of the United States is unmanufactured cotton, which is worth about 8 cts. a pound, and the value of which is 22% of the total exports of merchandise which amounted in a certain year to $863,200,487; how many pounds of cotton were exported? (Answer correct to 1,000,000.)

33. If the total value of merchandise imported into the United States in a certain year was $731,969,965, 13% being coffee of which there were 652,000,000 lbs.; what was the average price per pound of coffee?

34. The cost of collecting the customs revenue of $160,021,752 in a certain year being 4.52%, and of collecting the internal revenue of $146,762,865 being 2.62%, the former netted the government how much more than the latter?

35. The world's total production of wool in a certain year being 2,582,103,000 lbs., and the four largest producers being Russia with 290,000,000 lbs., Argentina with 280,000,000 lbs., the United States with 272,475,000 lbs., and Great Britain with 135,000,000 lbs., find what per cent of the total these countries produced, both collectively and separately. (Answer correct to 0.1%.)

36. The following table shows the receipts and certain disbursements of "old line" life assurance companies reporting to the N. Y. Insurance Department:

	Income.	Policies Paid.	Dividends to Policy Holders.	Expenses.
1871	$113,490,562	$28,773,041	$14,624,608	$20,242,707
1895	266,897,200	84,791,622	15,297,604	62,052,872

Find the rate of increase in each column.

37. Ascertain the population of the city or village in which you reside, according to the last three census reports; represent the statistics graphically and compute the rate of increase or decrease of population for each period.

38. Similarly for the average annual attendance of your school for the past five years.

39. The radius of the sun being 10,856% of that of the earth, the latter being 6370 km, compute the volume of the sun in cubic kilometers.

CHAPTER XIV.

Commercial Discounts and Profits.

It is the custom of manufacturers, publishers, and wholesale dealers to fix a price for their products and then to allow a discount under certain conditions.

E.g., a book may be published at \$2.00 with a discount of 25% to dealers, the book costing them \$2.00 — 25% of \$2.00, or \$1.50. The \$2.00 is known as the *list price*, the \$1.50 as the *net price*.

It frequently happens that wholesale houses issue expensive catalogues in which prices are specified. But as the cost of production varies, these prices change, and in order not to issue a new catalogue a house will print a new list of discounts for its customers. In some lines, indeed, the catalogue price has been so long fixed as to be several hundred per cent above the actual price, the latter being fixed by the discounts, of which there are often several.

E.g., paper bags are quoted at a certain price, but the bill sent to the retailer may read "Less 70% 25% 10%, 30 da., and 2% off in 10 da." This means that they can be produced so much cheaper than formerly that the purchaser is allowed a discount of 10%, then 25% from that price, then 70% from that, and finally, if he pays within 10 days instead of waiting 30 days, he is allowed a further discount of 2%. Hence, if the list price was \$100, the net price would be

\$90	after	deducting	10%	of	\$100.00,
67.50	"	"	25%	"	90.00,
20.25	"	"	70%	"	67.50,
19.84	"	"	2%	"	20.25.

In this case, the catalogue price is over 500% of the actual price paid. In Ex. 6 it is shown that it is immaterial in what order these discounts are taken.

On the bill heads of wholesale houses there is usually a note showing what discounts, if any, are allowed. For example, "Terms: 30 da. net, 1% 10 da."; "Terms: 60 da., or 2% if paid within 10 da."; "Terms: Net 60 da., or 2% disct. if paid in 10 da."

Illustrative problems.

I. The list price of some goods is \$62.70, a discount of 10% 6% 3% being allowed; required the net price.

Solution. $0.97 \cdot 0.94 \cdot 0.90$ of \$62.70 = \$51.45.

Analysis. 1. Let $l =$ list price, \$62.70.

2. Then $l - 0.10\,l = 0.90\,l$, the remainder after the first discount.

3. Then $0.90\,l - 0.06 \cdot 0.90\,l = 0.94 \cdot 0.90\,l =$ the remainder after the second discount.

4. Similarly, $0.97 \cdot 0.94 \cdot 0.90\,l =$ the remainder after the third discount = net price.

Application of logarithms. If the student has studied Chap. XI, this furnishes an application, the answer requiring no more than four figures and thus coming within the range of the table on pp. 114, 115.

$$\begin{aligned} \log\ 0.97 &= 0.9868 - 1 \\ \log\ 0.94 &= 0.9731 - 1 \\ \log\ 0.90 &= 0.9542 - 1 \\ \log 62.70 &= \underline{1.7973 \qquad} \\ \log 51.45 &= 1.7114 \end{aligned}$$

Unless logarithms are used, which is not advisable in practice, it is, of course, better to take 10% of \$62.70 and subtract, then 6% of this remainder and subtract, and then 3% of this remainder and subtract, than to perform the multiplication by $0.97 \cdot 0.94 \cdot 0.90$.

II. A merchant sells goods at a discount of 25% from the marked price and still makes a profit of 25% on the cost; at what per cent above cost did he mark them?

1. Let $c =$ the cost, and $m =$ the marked price.
2. Then $m - 0.25\,m = c + 0.25\,c$.
3. $\therefore\ 0.75\,m = 1.25\,c$.
4. $\therefore\ m = \dfrac{1.25\,c}{0.75} = 1.66\frac{2}{3}\,c$.
5. $\therefore$ he must mark them $66\frac{2}{3}\%$ above cost.

Exercises. 1. The list prices and rates of discount being as follows, find the cost:

	List Price.	Rates of Discount.
(a)	\$1271.50	$33\frac{1}{3}\%$, often called "a third off."
(b)	3.00	25%, " " "a quarter off."
(c)	125.00	15%.
(d)	37.50	20% $12\frac{1}{2}\%$ 6%.
(e)	2107.50	30% 8%.
(f)	403.80	25% 10% 4%.
(g)	3462.95	10% 3%.
(h)	178.65	$12\frac{1}{2}\%$ 8% 2%.
(i)	83.90	15% 7% 3%.
(j)	623.30	8% 2% 1%.
(k)	375.00	25% 10% 6%.
(l)	150.00	a third off 40%.

2. In each case of Ex. 1, suppose the buyer had sold the articles at the list price, what would have been his rate of gain on the cost?

3. In Ex. 1, what one rate of discount would have been equivalent to the several rates mentioned in (d), (e), ⋯⋯ (l)?

4. Suppose a dealer buys goods at "a third off" and sells them at "a quarter off" the list price, what is his rate of gain on the cost?

5. Show that the discounts 10% 8% 3% are equivalent to the discounts 3% 8% 10%, but not to the single discount 10% + 8% + 3%.

6. Generalizing Ex. 5, show that the discounts $r_1\%$ $r_2\%$ $r_3\%$ are equivalent to the discounts $r_3\%$ $r_2\%$ $r_1\%$, or $r_2\%$ $r_3\%$ $r_1\%$, etc.; that is, that it is immaterial in what order the discounts are taken.

7. The cost of certain goods and the rates of discount being as follows, find the list prices:

	Cost.	Rates of Discount.
(a)	\$1827.40	$12\frac{1}{2}\%$.
(b)	436.90	25% 10%.
(c)	49.63	30% 12% 6%.
(d)	2341.50	30% 10% 2%.
(e)	693.49	25% 10% 3%.
(f)	127.90	$33\frac{1}{3}\%$ 6%.
(g)	647.00	20% 8% 1%.

8. A merchant buys goods listed at \$250, on which a discount of 15% 10% 3% is allowed; he sells the goods for \$225; what rate of profit does he make on the cost?

9. Prove that if the rates of discount are r_1, r_2, the equivalent single rate of discount is $r_1 + r_2 - r_1 \cdot r_2$, and hence that the single rate equivalent to *two* rates of discount equals their sum minus their product.

10. A bill of merchandise amounting to $327.50 was bought Oct. 1, "Terms: 3 mo. or 5% off 60 da., *or* 10% off 30 da." How much money would settle the bill Jan. 1? Nov. 20? Oct. 27?

11. What is the cost of a bill of hardware amounting to $1027.40, discounts 40% 10% 3%, freight being $10.60?

12. What is the net value of one case of prints containing 3000 yds. @ 6 cts. per yd., less 8% discount; package $0.40, freight $0.95?

13. A merchant buys goods at a discount of 30% 20% and sells them at $\frac{1}{8}$ off the list price; what is his gain per cent on the cost?

14. Which is the better for the buyer to take, a discount of

(*a*)	30% 15% 10%	or one of	47%?
(*b*)	10% 10% 5%	"	23%?
(*c*)	15% $12\frac{1}{2}$%	"	$25\frac{5}{8}$%?
(*d*)	12% 8% 1%	"	19%?
(*e*)	10% 6% 2%	"	17%?
(*f*)	30% 30% 30%	"	66%?

15. At what per cent above cost must goods be marked in order to take off from the marked price

(*a*)	10%	and still make	a profit of	8%	on the cost?
(*b*)	25%	"	"	20%	"
(*c*)	20%	"	"	30%	"
(*d*)	10%	"	"	50%	"
(*e*)	r_1%	"	"	r_2%	"
(*f*)	15%	and lose		5%	"
(*g*)	20%	"		15%	"
(*h*)	r_1%	"		r_2%	"
(*i*)	30%	and neither gain nor lose?			

16. A dealer purchases some goods listed at $281.50, $\frac{1}{8}$ off and 5% for cash; if he pays on delivery, what is the net price, and what per cent discount can he give on the list price in order to make a profit of 15%?

17. Three rates of discount, 10% 10% r% are equivalent to the single rate 27.1%; find r.

18. What are the three equal rates of discount equivalent to the single rate 48.8%?

CHAPTER XV.

Interest, Promissory Notes, Partial Payments.

I. Simple Interest.

THERE is practically a single type of problem in this subject, *given the principal, rate, and time, to find the interest.* Bankers and others who frequently meet this problem find the interest by the help of printed tables. These tables are usually based on 360 days to the year, and in using them it is customary to reckon exact days between dates. Some banks use tables based on 365 days to the year, this being the fairer method although yielding less interest.

People generally, working without tables, reckon 360 days to the year, but they find the difference between dates by subtracting months and days, calling 30 days 1 month, and this is the method used in this work.

Thus, to find the time from July 2 to Sept. 2, it is customary, if one has no interest table, to say

9 mo.	2 da.
7 "	2 "
2 mo.	

But if an interest table is at hand, it will readily appear that

Sept. 2 is the 245th day of the year,
and July 2 " 183d " "
and that the difference in time is 62 da.

For exercises in analysis certain other problems are usually given in school, though rarely met in business. A few such have been inserted, types appearing in Problems II, III, IV, VI, on p. 138.

Interest is reckoned as a certain per cent of the principal. When a rate is specified, the words "for one year" are to be understood unless the contrary is stated.

I.e., if it is said that the rate of interest is 6%, it means that the interest for 1 yr. is 6% of the principal. Occasionally, however, interest is quoted by the month, as 1% a month.

Interest table. The following represents the first part of a page from an interest table such as bankers use.

This particular page, if printed in full, would give the interest at 6% for 3 mo. and any number of additional days from 0 to 29. This portion gives only 0, 1, 2, 3 da. in excess of 3 mo., this being sufficient for illustration.

3 MONTHS. 6%.

TOTAL DAYS.	1000	2000	3000	4000	5000	6000	7000	8000	9000	DAYS OVER 3 MO.
90	15.00	30.00	45.00	60.00	75.00	90.00	105.00	120.00	135.00	0
91	15.17	30.33	45.50	60.67	75.83	91.00	106.17	121.33	136.50	1
92	15.33	30.67	46.00	61.33	76.67	92.00	107.33	122.67	138.00	2
93	15.50	31.00	46.50	62.00	77.50	93.00	108.50	124.00	139.50	3
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

The method of using the table will be seen from the following computation of the interest on $3975 for 93 da. at 6%:

Int. on	$3000 =	$46.50	
" "	900 =	13.95,	$\frac{1}{10}$ of int. on $9000.
" "	70 =	1.09	
" "	5 =	.08	
		$61.62	

It will be noticed that the interest on ordinary sums can be told by merely glancing at the table. Thus, the interest on $250 for 3 mo. is $3.00 + $0.75 = $3.75.

Illustrative Problems.

I. What is the interest on \$360 for 1 yr. 6 mo. 10 da. at 6%?

1. 1 yr. 6 mo. 10 da. $= 1\frac{19}{36}$ yrs.
2. Int. for 1 yr. $= 6\%$ of \$360.
3. $\therefore$ int. for $1\frac{19}{36}$ yrs. $= \frac{55}{\cancel{36}} \cdot 0.06 \cdot \$\overset{10}{\cancel{360}} = \$33.$

II. The interest on \$360 for 1 yr. 6 mo. 10 da. is \$33; required the rate.

1. 1 yr. 6 mo. 10 da. $= 1\frac{19}{36}$ yrs. $= \frac{55}{36}$ yr.
2. $\because$ the int. for $\frac{55}{36}$ yr. $= \$33,$
   $\therefore$ " " " 1 yr. $= \$33 \div \frac{55}{36} = \frac{36}{55} \cdot \$33.$
3. $\because r\%$ of \$360 $= \frac{36}{55} \cdot \$33,$

$$\therefore r\% = \frac{\cancel{36}}{\underset{5}{\cancel{55}}} \cdot \frac{\$\overset{3}{\cancel{33}}}{\$\underset{10}{\cancel{360}}} = 6\%.$$

III. How long will it take the interest on \$360 at 6% to equal \$33?

1. The int. for 1 yr. $= 6\%$ of \$360.
2. $\therefore$ " " t yrs. $= t \cdot 6\%$ of $\$360 = \$33.$
3. $\therefore \quad t = \frac{\$33}{0.06 \cdot \$360} = 1\frac{19}{36}.$
4. $\therefore$ it will take $1\frac{19}{36}$ yrs., or 1 yr. 6 mo. 10 da.

IV. On what sum of money will the interest for $1\frac{19}{36}$ yrs. at 6% equal \$33?

1. Let $p =$ the principal.
2. $\because$ the int. for $1\frac{19}{36}$ yrs. at 6% on p is \$33,
3. $\therefore 1\frac{19}{36} \cdot 6\% \cdot p = \$33.$
4. $\therefore \quad p = \frac{\$33}{1\frac{19}{36} \cdot 0.06} = \$360.$

V. Find the amount (principal plus interest) of \$360 for 1 yr. 6 mo. 10 da. at 6%.

1. By Prob. I, amt. $= \$360 + \frac{55}{36} \cdot 0.06 \cdot \$360$
   $= (1 + \frac{55}{36} \cdot 0.06) \cdot \$360 = \$393.$

VI. What principal will amount to \$393 in $\frac{55}{36}$ yr. at 6%?

1. Rate for $\frac{55}{36}$ yr. $= \frac{55}{36} \cdot 0.06.$
2. Let $p =$ the principal; then,
   $p + \frac{55}{36} \cdot 0.06\, p = \$393,$ or
   $(1 + \frac{55}{36} \cdot 0.06)\, p = \$393.$
3. $\therefore p = \frac{\$393}{1 + \frac{55}{36} \cdot 0.06} = \$360.$

Exercises. 1. Find the interest on $10,000 from July 2 to Sept. 2, at 6%, reckoning the time as follows:

(*a*) Subtract the months and days, calling 30 da. = 1 mo., and 360 da. = 1 yr.

(*b*) Take exact days between dates, but let 360 da. = 1 yr.

(*c*) Subtract as in (*a*), but let 365 da. = 1 yr.

(*d*) Take exact days between dates, but let 365 da. = 1 yr.

2. Of the four plans given in Ex. 1,

(*a*) Which gives the most interest?

(*b*) Which is easiest without interest tables?

(*c*) Which is the fairest?

(*d*) Which result differs most from the fairest result?

(*e*) Why do people generally, without interest tables, use method (*a*)?

(*f*) Why do bankers generally use method (*b*)?

(*g*) Why is method (*c*) not used?

(*h*) Why do the government and some banks use method (*d*)?

3. Find the interests on the following principals for the times and at the rates specified:

(*a*) $250 for 2 yrs. 4 mo. 8 da. at 6%.
(*b*) $40 " 1 " 3 " " 5%.
(*c*) $125 " 8 " 15 " " 7%.
(*d*) $350 " 3 " 6 " " 4%.
(*e*) $820 " 6 " " $4\frac{1}{2}$%.
(*f*) p " t " " r%.

4. Find the rates at which the following principals will yield the interests mentioned in the respective times:

(*a*) $300 yields $45 interest in 2 yrs. 6 mo.
(*b*) $175 " $10.50 " 1 " 6 "
(*c*) p " i " t "

5. Find the times in which the following principals will yield the interests mentioned at the respective rates:

(*a*) $450 yields $24 at 6%.
(*b*) $125 " $6.25 " 4%.
(*c*) p " i " r%.

6. Find the principals which will yield the following interests at the times and rates mentioned:

(*a*) $62.50 interest in 1 yr. 3 mo. at 4%.
(*b*) $5 " 1 " 1 " 10 da. " 6%.
(*c*) i " t " " r%.

7. Find the principals which will amount to the following sums at the times and rates specified:

(a)	\$280	in 2 yrs.		at 6%.
(b)	\$45.25	" 1 "	10 mo. 15 da.	" 7%.
(c)	a	" t "		" r%.

8. From the portion of the interest table given on p. 137, find the interest at 6% on:

(a)	\$275	for 3 mo.	
(b)	\$750	" 3 "	3 da.
(c)	\$9275	"	91 "
(d)	\$5750	"	93 "

Short methods. Before interest tables were common, short methods of computing interest were valuable. At present those who have much work of this kind use these tables. One method is, however, of enough value to be mentioned, especially as the most common rate of interest is 6%, and as most notes run for 90 days or less.

Required the interest on \$250 for 63 da. at 6%.

1. ∵ the rate for 1 yr. is 6%,
2. ∴ " $\frac{1}{6}$ ", or 2 mo., is 1%.
3. 1% of \$250 = \$2.50, interest for 60 da.
4. $\frac{1}{20}$ " \$2.50 = \$0.12½, " 3 "
5. ∴ \$2.62½ = " 63 "

In practice it is merely necessary to put down these figures, the vertical line representing the decimal point:

\$2	50	60 da.
	12½	3 "
\$2	62½	63 da.

For 7%, add $\frac{1}{6}$ of \$2.62½, and similarly for other rates.

Exercises. Find the interests on the following sums for the times and at the rates specified:

1. \$144, 30 da., 6%.
2. \$750, 93 " 6%.
3. \$125, 60 " 6%.
4. \$250, 93 " 6%.
5. \$400, 33 " 7%.
6. \$50, 90 " 5%.
7. \$150, 60 " 8%.

II. Promissory Notes.

Most promissory notes between individuals are of substantially the following form :

$500. Chicago, Ill., Dec. 3, 1900.

Thirty days after date, I promise to pay to John Jones, or order, five hundred dollars, value received, with interest at 5%. John Smith.

1. In this case John Smith is the *maker*, John Jones the *payee*, $500 the *face*, and the face plus the interest is the *amount*, or *future worth*.

2. This note is *negotiable*, and may be sold by the payee, the transfer being indicated by his *indorsing* the note, that is, by writing his name across the back. Notes payable to the payee "or bearer" are also negotiable.

3. By indorsing the note the payee becomes responsible for its payment in case the maker does not pay it. But if the buyer is willing to take the note without this guarantee, the indorser may be released by first writing the words "Without recourse" across the back, and then his name.

4. The indorsement may be made *in blank*, that is, the payee may merely write his name across the back, or *in full*, that is, the payee may specify the person to whose order it is to be paid, thus:

"Pay to the order of John Brown.
John Jones."

5. A note *matures* on the day when it is legally due. When the time is specified in days, exact days are counted in ascertaining maturity; when in months, calendar months are counted.

6. Many states, following an old custom, allow three *days of grace* for the payment of notes. That is, a note dated Dec. 3, the time being "30 days after date," is legally due Jan. 2 + 3 days, or Jan. 5, a fact indicated by writing "Due Jan. 2/5." A considerable number of states have abolished these days of grace, and the custom will in time become obsolete. Where the law still allows them it is quite common for notes to bear the words "Without grace."

7. The law as to the time of payment of notes due on legal holidays varies in different states.

8. If a note reads "with interest," but does not specify the rate, it draws the rate specified by the law of the state. If it does not call for interest, it draws none until it is due and payment is demanded, after which it draws the legal rate.

9. In some states the law specifies what is called the "legal rate," and then specifies a maximum rate above which no contract for interest is legal. In other states the "legal rate" is also the maximum. Some states specify no maximum, allowing the parties to the contract to fix any rate they wish. Interest above the maximum rate allowed by law is called *usury*, and the taking of usury is punished according to the laws of the various states in which it is forbidden.

Notes payable at a bank are discussed in Chap. XVI on Banking Business. The protest of notes is also discussed in that connection.

Exercises. 1. In your state, are days of grace allowed on promissory notes? When are notes which mature on legal holidays payable in your state?

2. What is the "legal rate" of interest in your state? Is there a maximum rate beyond this? What is the punishment for usury in your state?

3. What is the rate at which money is usually loaned to responsible persons in your vicinity?

4. Write a 90-day note, signed by Peter Brown and payable to your order, bearing the rate which you found in Ex. 3; indorse it so that it shall be payable to the order of Robert Jones.

5. Find the dates of maturity of, the interests on, and the amounts of promissory notes for the following sums, at the specified rates, supposing the notes paid when due; add the days of grace if such is the law in your state, otherwise not:

(*a*)	\$500,	dated	Feb. 7,	due	6 mo.	after date, at	6%.
(*b*)	\$250,	"	Mar. 1,	"	1 yr. 6 mo.	"	5%.
(*c*)	\$100,	"	July 15,	"	90 da.	"	7%.
(*d*)	\$750,	"	Sept. 7,	"	2 yrs.	"	$4\frac{1}{2}\%$.
(*e*)	\$1275,	"	Aug. 10,	"	60 da.	"	6%.
(*f*)	\$350,	"	June 3,	"	4 mo.	"	at the rate found in Ex. 3.
(*g*)	\$50,	"	Dec. 10,	"	2 "	"	"
(*h*)	\$200,	"	Oct. 5,	"	4 "	"	"

III. Partial Payments.

If a note or other obligation draws simple interest, and partial payments have been made at various times, the sum due at any specified date is usually computed as follows:

1. The interest on the principal is found to that time when the payment or payments which have been made equal or exceed this interest.

2. The payment or payments are then deducted and the remainder is treated as a new principal.

These directions constitute what is known as the United States Rule of Partial Payments, the only legal method in most states. A few states, however, require other methods, and in these the teacher should explain the law and require the problems solved accordingly.

The United States Rule, and the reason for the first sentence, will be understood from a single problem: A note for $1000 is dated Jan. 2, 1900, and draws 6% interest; the following payments have been made, — Jan. 2, 1901, $1; July 2, 1901, $89; Sept. 2, 1901, $500; required the amount due Jan. 2, 1902.

1. On Jan. 2, 1901, the $1000 amounts to $1060.
2. If the $1 were now deducted the new principal would be $1059.
3. But then the borrower would be paying interest on $59 more than he agreed.
4. ∴ it would not be right to deduct the $1, or any other sum which might be paid, unless it should equal at least $60.
5. The practical solution usually appears in the following form:

1901	7 mo.	2 da.				$1000.
1900	1 "	2 "	Int. to July 2, 1901,			90.
1	6		Amt.	"		$1090.
			Paymts.	"	($1 + $89),	90.
	9	2	New prin.	"		$1000.
	7	2	Int. to Sept. 2, 1901,			10.
	2		Amt.	"		$1010.
			Paymt.	"		500.
1902	1	2	New prin.	"		$510.
1901	9	2	Int. to Jan. 2, 1902,			10.20
	4		Amt.	"		$520.20

Payments less than the accrued interest are seldom made. When they are made it is usually possible to detect the fact that they are less than the interest, before computing the amount due on that date.

Partial payments are usually indorsed on the note, that is, written across the back, as, for example:

"Jan. 2, 1902. Rec'd \$10.
July 5, 1902. Rec'd \$50."

Exercises. Find the amounts due at the dates of settlement specified:

DATE OF NOTE.	FACE.	RATE.	PARTIAL PAYMENTS.	SETTLED.
1. Jan. 10	\$603	6%	June 1, \$100; Aug. 15, \$50; Sept. 20, \$30	Oct. 15
2. Apr. 4, 1900	\$125	7%	Aug. 1, \$5; Oct. 16, \$30; Jan. 10, 1901, \$75	Apr. 1, 1901
3. Nov. 1, 1899	\$50	6%	Dec. 12, \$5; Jan. 10, 1900, \$40; Feb. 1, \$1	Apr. 10, 1901
4. Jan. 11, 1899	\$500	5%	Jan. 11, 1900, \$20; July 11, \$15; Jan. 11, 1901, \$50	Mar. 5, 1901
5. June 14, 1897	\$375	6%	Sept. 10, \$3.50; Nov. 15, \$4.75; Jan. 7, 1898, \$51.75; Jan. 11, 1899, \$200	June 21, 1899
6. May 1, 1897	\$1000	5%	Sept. 1, \$5; Nov. 1, \$3; Mar. 1, 1898, \$100; July 15, \$275	Sept. 1, 1898
7. Apr. 1, 1901	\$200	$4\frac{1}{2}$%	July 1, \$50; Jan. 16, 1902, \$5; June 10, 1902, \$2; July 1, 1902, \$75	Sept. 1, 1902
8. June 1, 1898	\$800	$5\frac{1}{2}$%	Jan. 3, 1899, \$35; Aug. 1, \$15; Nov. 3, 1899, \$70; Aug. 16, 1900, \$100; Feb. 1, 1901, \$125; Sept. 1, 1902, \$180	Dec. 1, 1902
9. Apr. 1, 1901	\$500	6%	Jan. 1, 1902, \$100; Aug. 7, 1903, \$25	Jan. 1, 1904
10. Jan. 1, 1900	\$2000	6%	Jan. 1, 1901, \$500; Apr. 1, 1902, \$250; Dec. 16, 1903, \$100; Jan. 1, 1905, \$600	Nov. 13, 1905
11. Feb. 5, 1904	\$675	5%	Apr. 1, 1905, \$25; Aug. 5, 1905, \$100; Sept. 5, 1905, \$50; Jan. 20, 1906, \$200	Jan. 20, 1907
12. May 2, 1900	\$575	5%	July 1, 1901, \$75; Sept. 3, 1901, \$100; Jan. 1, 1902, \$50; Apr. 1, 1902, \$100; July 1, 1902, \$100; Sept. 17, 1903, \$50	Sept. 17, 1904

IV. Compound Interest.

Savings banks usually add the interest to the principal at the end of the interest period, say every six months. The whole amount then draws interest, the depositor thus receiving interest on interest, or *compound interest.* Otherwise, the subject is not often met in a practical way, although banks, by loaning their interest as it is received, really have all of the benefits of compound interest.

As in simple interest, there is a single case of practical value — *given the principal, rate, and time, to find the compound interest or the amount.*

E.g., what is the amount of \$500 for 3 yrs. at 3% compound interest?

1. The amt. of \$500.00 and int. for 1 yr. = \$515.00.
2. " \$515.00 " " = \$530.45.
3. " \$530.45 " " = \$546.36.
4. ∴ " \$500.00 " 3 yrs. = \$546.36.

Similarly, what is the amount of \$150 for 3 yrs. at 4%, interest compounded semi-annually?

1. The amt. of \$150 and int. for 6 mo. $= \$150 + 0.02 \cdot \$150$
$= 1.02 \cdot \$150.$
2. ∴ " $1.02 \cdot \$150$ " $= 1.02 \cdot 1.02 \cdot \$150$
$= 1.02^2 \cdot \$150.$
3. ∴ " $1.02^2 \cdot \$150$ " $= 1.02^3 \cdot \$150.$
4. And finally, the amount for 6 six-month periods
$= 1.02^6 \cdot \$150 = \$168.93.$

Interest tables. While compound interest is not in general use, it frequently happens that large investors wish to compute the amount resulting from reinvesting all interest as it becomes due; in other words, they wish to ascertain the amount of a certain sum at compound interest. For this purpose they resort to compound-interest tables, a specimen of which is given on p. 146. A table of logarithms evidently answers the same purpose.

AMOUNT OF $1000 AT COMPOUND INTEREST.

YEARS.	2%	$2\frac{1}{2}$%	3%	4%	5%	6%
1	1020.00	1025.00	1030.00	1040.00	1050.00	1060.00
2	1040.40	1050.63	1060.90	1081.60	1102.50	1123.60
3	1061.21	1076.89	1092.73	1124.86	1157.63	1191.02
4	1082.43	1103.81	1125.51	1169.86	1215.51	1262.48
5	1104.08	1131.41	1159.27	1216.65	1276.28	1338.23
6	1126.16	1159.69	1194.05	1265.32	1340.10	1418.52
⋮	⋮	⋮	⋮	⋮	⋮	⋮

If the interest is at the rate of 4%, 5%, or 6% per year, but compounded semi-annually, the amount is evidently the same as if the rate were 2%, $2\frac{1}{2}$%, or 3%, respectively, compounded annually for a period twice as long.

E.g., what is the amount of $2750 for 3 yrs. at 5%, compounded semi-annually?

Amt. of $1000 for 6 yrs. at $2\frac{1}{2}$% compounded annually = $1159.69.
" $2750 = 2.75 × $1159.69 = $3189.14.

The subject of compound interest is still further discussed in the Appendix, Note III.

Exercises. Find the amounts of the following sums for the times and rates of compound interest specified:

1. $50, 2 yrs. 6 mo., 3%, compounded semi-annually.
2. $168, 4 yrs. 3 mo., $4\frac{1}{2}$%, " annually.
3. $1200, 3 yrs. 2 mo., 4%, " semi-annually.
4. $350, 1 yr. 8 mo., 4%, " quarterly.
5. $*p*, 1 yr., r%, " annually; also for 2 yrs., 3 yrs., t yrs.

6. From Ex. 5, find p, the principal, which at r%, compounded annually for t yrs., amounts to a.

7. From Ex. 6, find p, given $a = \$123.73$, $t = 3$, $r\% = 4\%$.

8. From the compound-interest table, find the amount of $500 at 4%, compounded annually for 5 yrs.

9. Also of $2500 at 5%, compounded semi-annually for 2 yrs.

10. Also of $350 at 3%, compounded annually for 6 yrs.

11. Also of $4000 at 2%, compounded annually for 4 yrs.

V. Annual Interest.

In some states, if a note or bond contains the words "with interest payable annually" this interest, if left unpaid, also draws interest to the day of settlement, or until cancelled by payment. The note or bond is then said to draw *annual interest.*

E.g., to find the amount due on a $500 note dated Jan. 1, 1900, drawing annual interest at 6%, no payments made until the day of settlement, Jan. 1, 1904.

1.	Face of note	= $500.
2.	Int. on $500 for 4 yrs. at 6%	= 120.
3.	Int. on $30 for 3 yrs. + 2 yrs. + 1 yr. at 6%	= 10.80
4.	Amt. due Jan. 1, 1904	= $630.80

Unless annual interest is allowed in the state in which this book is used, this subject may be omitted.

Semi-annual or quarterly interest is treated in a similar manner.

Exercises. 1. What is the amount due at the end of 3 yrs. on a $1000 note bearing annual interest at 5%, no payments having been made?

2. In the western states *coupon notes* are often given, that is, notes bearing coupons which are themselves promissory notes for the interest due, and also drawing interest, often at a higher rate. Find the amount of a coupon note for $1000 at the end of 5 yrs., the principal drawing 6%, the coupons representing the interest due annually and drawing 8% remaining unpaid.

3. A coupon note draws 6%, the coupons being due semi-annually and drawing 10% if unpaid; the face of the note being $800, and no payments having been made, find the amount due at the end of 4 yrs.

4. What is the amount due at the end of 4 yrs. on a $300 note bearing 5% interest, payable semi-annually, no payments having been made?

5. A coupon note draws 6%, the coupons being due semi-annually and drawing 8%, if unpaid; the face of the note being $500, and no payments having been made, find the amount due at the end of 3 yrs.

6. In Ex. 5, supposing the first three coupons had been paid when due, find the amount due at the end of 3 yrs.

CHAPTER XVI.

Banking Business.

THE ordinary business of a bank is largely included under the following heads:

1. Receiving deposits and paying from the same on presentation of checks signed by the depositor.

2. Lending money upon promissory notes or (chiefly in the case of savings banks) upon bonds and mortgages.

3. Discounting notes which individuals may own and upon which they wish to realize money before the notes are due.

4. Selling drafts on other banks, and collecting drafts drawn by one person or corporation on another. (See Chap. XVII.)

I. DEPOSITS AND CHECKS.

If a person deposits money in a savings bank (or in the savings department of a bank having both savings and commercial departments), he usually receives a book in which are written the sums deposited and drawn out. If he wishes to draw out any money, he presents his book for the debiting of the amount and is usually required to sign a receipt. Savings banks usually pay from 2% to 4% interest compounded semi-annually.

Ordinary deposits in other banks do not draw interest, the deposit being made for convenience and safety. When the depositor wishes to draw upon his deposit, he makes out a check, of which the following is a common form:

Chicago, Ill., ____________________ No. ______

First National Bank of Chicago.

Pay to the order of

______________________________ $ ________

______________________________ Dollars.

____

A check is usually made payable to:

1. "Self," in which case the drawer alone can collect it.

2. The payee or bearer, or merely to "Bearer," in which cases any one can draw the money.

3. The order of the payee, in which case the payee must indorse it.

Most banks also receive money and issue Certificates of Deposit, of which the following is a common form:

MICHIGAN.

No. *14762* **Certificate of Deposit.**

FIRST NATIONAL BANK OF DETROIT.

Detroit, Mich., January 2, 1900.

John Doe *has deposited in this Bank*

Five hundred —————————— *Dollars*

payable to his order in current funds on the return of this Certificate properly endorsed, with interest at the rate of 3 per cent per annum if left three months, or 4 per cent if left six months. Interest hereon will cease one year from date.

\$500.00 *Wm. Smith, Teller.* *Richard Roe, Cashier.*

Exercises. 1. Write a check for \$54.75.

2. Also one payable to the order of yourself, and properly indorse it so that it can be collected only by Richard Roe or his order.

3. Write a certificate of deposit payable to your order, for \$75, dated Jan. 4, drawing 3% if left 3 mo., or $3\frac{1}{2}$% if left 6 mo. Compute its value on Nov. 19; also on May 11; also on Mar. 23.

II. Lending Money.

If a person wishes to borrow money from a bank, and the bank is willing to lend it to him, he usually gives a promissory note. This note may be secured by depositing with the bank some evidences of value, as stocks, bonds, etc., usually known as "Collateral," or by having some responsible person indorse the note. At present banks frequently loan money without an indorser to persons of unquestionable financial standing, a custom formerly not common. Since the borrowing of money on an indorsed promissory note is the method most commonly followed, this is the only one here discussed.

A bank note is usually in the following form:

> Boston, Mass., May 5, 1899
>
> Two months after date, without grace, I promise to pay to the order of Richard Roe $ 75.00
>
> Seventy-five ——————————— Dollars
>
> at The First National Bank, Boston, Mass.
>
> Value Received,
>
> Residence 4030 Beacon St. John Doe.
>
> Due July 5.

Such notes are usually made payable in 1, 2, or 3 mo., or in 30, 60, or 90 da., so that the bank can get its interest often, the interest then being reloaned. It was formerly the general custom to add three "days of grace," as mentioned on p. 141; but as already stated, a considerable number of states have abolished this custom. In the above note the words "without grace" make the note mature July 5; otherwise it would mature July 8, drawing interest to that date.

As a rule no interest is specified in such notes, but interest is paid in advance and is called *discount*. The bank thus gets interest on interest, but this is allowed, in such cases, by law.

In the case of the above note, John Doe, the *maker*, wishes to borrow $75.00. He makes the note payable to the order of Richard Roe, with whose financial standing the bank is satisfied. Richard Roe indorses it, thereby promising to pay it if John Doe does not. The maker then takes it to the bank and receives $75.00 less the interest (or *discount*) on $75.00 for 2 mo. at the usual rate. Since Richard Roe indorses this note for the accommodation of the maker, he is called an *accommodation indorser.*

On July 5, if John Doe does not pay this note, the bank places it in the hands of a Notary Public, who sends to Richard Roe, the indorser, a Notice of Protest. If this is not sent promptly, the indorser may assume that the note has been paid, and he is released by law. If this notice is placed in a properly addressed sealed envelope and deposited in the post office by the notary, the demands of the law have been fulfilled. The law of protest varies, however, in different states.

In discounting notes, banks count the time by months or days according as the note specifies, and then compute the interest by the help of tables usually based on 360 da. to the year, calling 30 da. 1 mo.

Exercises. 1. Are "days of grace" allowed by law in your state? (If so, always add them in solving the problems in this section; otherwise not.)

2. What is the day of maturity and the discount on the following notes:

	Date.	Time Named.	Face.	Rate of Discount.
(*a*)	Apr. 1,	60 da.,	$250,	6%.
(*b*)	Oct. 17,	3 mo.,	$5000,	5%.
(*c*)	May 10,	90 da.,	$125,	7%.
(*d*)	Dec. 12,	2 mo.,	$50,	6%.
(*e*)	July 7,	4 mo.,	$600,	5%.

3. What is the usual rate of discount on bank notes in your vicinity? Using that rate, find the discounts on the following notes:

	Date.	Time Named.	Face.
(*a*)	Apr. 15,	4 mo.,	$1000.
(*b*)	Jan. 3,	60 da.,	$500.
(*c*)	Aug. 5,	90 da.,	$750.
(*d*)	Dec. 9,	3 mo.,	$50.
(*e*)	Oct. 8,	2 mo.,	$75.

4. Write and properly indorse bank notes subject to the conditions stated in Ex. 3.

III. Discounting Notes.

Merchants frequently take notes from their customers, running 1, 2, or 3 mo. or even longer, and drawing interest. Such notes are often made payable at the bank in which the merchant keeps his account so that, in case he needs the money on a note before it is due, and sells it to the bank, the latter can the more easily collect it. In case of sale, the seller indorses the note and the bank *discounts* it; that is, the bank pays the sum due at maturity, less the discount (interest) on that sum, this difference being called the *proceeds.*

It will be seen that so far as the bank is concerned this process of discounting a note held by a customer is essentially that already described of lending money on an indorsed note. There are, however, two differences:

1. The indorser is not now an accommodation indorser; he is the owner of the note and he receives the money from the bank. If, however, the maker does not pay the note when it becomes due it is protested like any other note and the indorser is held responsible as explained on p. 151.

2. The note usually draws interest and it frequently is not discounted on the day of its date. The discount is therefore reckoned on the face of the note plus the interest, or on the *future worth*, for the time between the day of discount and the day of maturity. This time is occasionally computed in exact days, but more often in months and days.

E.g., a merchant takes from a customer a note for $755.50, dated Apr. 16, due in 90 da. without grace, at 6%. Needing the money on May 1 he indorses the note and deposits it in his bank. If the bank is discounting at 6%, it gives him credit for the proceeds. The computation is as follows:

Face of note	=	$755.50
Int. for 90 da.	=	11.33
Future worth	=	$766.83
Disc't for 75 da.	=	9.59
Proceeds	=	$757.24

Exercises. In the following problems take the rate of discount usually charged by banks in your vicinity, except as otherwise specified, allowing days of grace or not according to their custom. The first exercise includes the practical business problems; the rest are of value merely for the analysis.

1. Find the discount and the proceeds on the following notes:

	FACE.	DATE.	TIME TO RUN.	INTEREST.	DISCOUNTED.
(*a*)	$136.75,	Feb. 7,	3 mo.,	7%,	Apr. 1.
(*b*)	$75.50,	May 10,	60 da.,	6%,	June 2.
(*c*)	$352.00,	Oct. 5,	2 mo.,	6%,	Oct. 5.
(*d*)	$50.75,	July 8,	90 da.,	None,	Sept. 1.
(*e*)	$800.00,	Jan. 10,	4 mo.,	5%,	Feb. 10.
(*f*)	$62.25,	Dec. 8,	3 mo.,	None,	Dec. 27.

2. Find the face of the following notes:

	PROCEEDS.	DATE.	TIME TO RUN.		INT.	DISCOUNTED.	RATE OF DISCOUNT.
(*a*)	$75.24,	Feb. 4,	3 mo.,	without grace	6%,	Mar. 4,	7%.
(*b*)	$81.46,	July 13,	2 mo.,		5%,	July 20,	6%.
(*c*)	$101.56,	Oct. 3,	4 mo.,		7%,	Dec. 18,	6%.
(*d*)	$39.85,	Apr. 1,	2 mo.,		None,	Apr. 1,	6%.

3. For how long is a note for $74.60 discounted at 6%, if the proceeds are $73.85?

4. At what rate is a note for $125.50 discounted for 4 mo., if the discount is $2.09?

5. Do you know of any savings bank in your vicinity? If so, what rate of interest does it pay and how often is this compounded? Under these conditions, what would be the amount at the end of 3 yrs. of $100 invested July 1?

6. If you had a check on a bank in your vicinity, payable to your order, what would be the steps necessary to get the money? What would be the steps necessary to transfer it to another person so that the money could be drawn only on his order?

7. Find the difference in the amount of $100 invested in a savings bank at 4%, compounded semi-annually, and the amount of the same sum at simple interest at $4\frac{1}{2}$%, the time being 5 yrs. in each case; also for 6 yrs.; also for 7 yrs.

8. Which yields the better income on $100 in 10 yrs., 4%, compounded semi-annually, or 5% simple interest?

9. What principal will amount to $29,588.62 in 3 yrs. 3 mo. 3 da. at 6%?

CHAPTER XVII.

Exchange.

If a person in one place owes a debt in another, he can settle it in a variety of ways.

1. He may send the money

(*a*) By an unregistered letter; this is liable to be lost or stolen, although with our present postal service this liability is slight.

(*b*) By a registered letter; in case of loss this can be traced to the one at fault; in case of loss by accident no recovery is possible, but otherwise the government, while not holding itself responsible, requires the one at fault to make good the loss.

(*c*) By express or other messenger, in which case the company or messenger is liable for loss.

2. He may cancel the debt by sending

(*a*) A check on his home bank where he has money deposited, in which case the creditor may have to pay a bank for collecting it.

(*b*) A draft drawn by some bank on a bank in a large city like New York or Chicago; such a draft, especially if presented by a customer, is usually cashed without discount at any bank.

(*c*) A postal money order; this is not as safe as (*a*) or (*b*) since the government cannot be sued in case of payment to the wrong person; identification is required, however, unless the sender waives it.

(*d*) An express money order, issued by various express companies and costing the same as the postal order; in case of loss or of payment to the wrong person these companies can be sued.

(*e*) A telegraphic order; this method is the most expensive, but the most rapid.

The subject of Exchange relates to the second of these plans and includes the five methods named.

(*a*) **The check.** This instrument has already been described on p. 149.

If a check is drawn by John Doe of Albany on a bank in that city, payable to the order of Richard Roe of Cleveland, the latter on receiving it indorses it and deposits it in the bank where his account is kept. This bank will probably collect it for him without charge. This is the usual plan, and a large part of the indebtedness of the country is settled by checks.

If Richard Roe has no bank account, the bank to which he takes the above check will require his identification, will charge him *exchange*, that is, a small sum for collecting it, and will probably not pay him the money until it has been received from Albany.

(*b*) **The draft.** Drafts are usually in substantially this form :

First National Bank of Albany. No. 18701

Albany, N. Y., July 6, 1898.

Pay to the order of John Doe $53 75/100

Fifty-three and 75/100 —————— Dollars

To Mercantile National Bank,
New York City. A. B. Jones

Cashier.

It will be noticed that a draft is quite like a check, but it is signed by the cashier of some bank and is drawn on some bank in a large commercial center.

In the case mentioned under (*a*), John Doe might have purchased a draft for the amount of his indebtedness, payable to his own order or to the order of Richard Roe; if to his own order, he would have indorsed it payable to the order of Richard Roe. He might have to pay a slight premium to the bank, usually 10 cts. to 15 cts. for drafts under $100. On receipt of the draft, Richard Roe would indorse it and receive the money, usually without any discount, at a bank.

The drafts already described are sometimes known as *bankers' drafts* to distinguish them from *commercial drafts.* The latter are extensively used by merchants, though rather as a means of demanding and collecting payment for a debt through the agency of banks than as a system of domestic exchange.

The great majority of such drafts are substantially in the following form :

> Cleveland, Ohio, July 6, 1898 No. 2347
>
> At sight pay to the order of
>
> The Second National Bank, Cleveland $ 53.75
>
> Fifty-three and $\frac{75}{100}$ ———————— Dollars
>
> To John Doe, Albany, N. Y. } Richard Roe.

1. In the above case suppose John Doe has bought goods of Richard Roe to the amount of $53.75, say on 30 da. credit, and does not pay the bill when due. Roe may then make out a draft as above, payable to the order of his bank, and deposit it for collection.

2. The Cleveland bank would send it to some Albany bank, asking it to collect and remit.

3. The Albany bank would send a messenger to John Doe and demand payment. In some states 3 days of grace are allowed on a sight draft, though not in New York. In such case, or in case of a time draft (*i.e.*, a draft payable a certain number of days after sight), Doe writes "Accepted, July 8, 1898" (if that is the date) across the face and signs it; at the proper time it is again presented by a messenger from the bank and payment is demanded.

4. In case Doe declines to accept it, or to pay it if due immediately, the draft is returned to the Cleveland bank and Roe is notified; he must then take other means for payment.

5. If it is paid, the Albany bank remits by draft to the Cleveland bank, deducting a small amount for making the collection.

The fluctuation of exchange. For small sums, say for \$500 or less, New York or Chicago exchange always sells at a premium of about 0.1%. This is to pay the bank for its trouble and for the expense of shipping the money when its balance at New York or Chicago gets low. Banks usually buy New York or Chicago drafts at par, that is, at their face value, thus making no charge for cashing them.

But on large sums the rate of exchange varies. If the San Francisco banks owe the New York banks \$1,000,000, they must send that amount by express, an expensive proceeding. If a man in San Francisco at that time wished to buy a draft on New York for \$10,000 they would charge him more than usual because they would have to express that much more to New York. But if a man in New York wished to buy a draft on San Francisco he might buy it for \$9999 or less because they would get their money at once and the risk and expense of transmitting it would be saved.

The premium or discount is usually quoted as a certain per cent of the face of the draft, but sometimes as the amount on \$1000. Thus, the quotation of $\frac{1}{4}$% premium is the same as that of \$2.50 premium.

Exercises. 1. New York banks are selling drafts on New Orleans banks at 0.1% discount; which city is owing the other the more money? How much would a draft on New Orleans for \$5000 cost in New York?

2. Denver is selling drafts on Boston at $\frac{1}{5}$% premium; which city is owing the other the more money? How much would a draft on Boston for \$15,000 cost in Denver?

3. Suppose the balance of trade between Cincinnati and Chicago is said to be largely against Cincinnati, what does this mean? In which place would the drafts on the other certainly be at a premium?

4. Suppose that in Denver drafts on New York are selling at $\frac{1}{5}$% premium, on Chicago at 0.1% premium, on San Francisco at 0.1% discount; what is the probable balance of trade between Denver and each of the other cities?

The clearing house. If the draft shown on p. 155 is indorsed by John Doe payable to the order of Richard Roe of Cleveland, Roe will take it to his Cleveland bank to be cashed or placed to his credit. The Cleveland bank will send it to the New York bank with which it does business, say the Chemical National, and will receive credit for it. The next morning the Chemical National will send it to the Clearing House, where the leading New York banks send representatives to transfer the drafts held by each on the others. There it goes to the representative of the Mercantile National Bank on which it is drawn, and is paid. By the Mercantile National it is finally returned to the First National Bank of Albany which drew it.

Since both the Chemical and Mercantile National Banks have drafts on one another, and so for all other banks in the Clearing House, only a comparatively small balance is necessary to settle all accounts.

Every large city has its own clearing house, but the one at New York is the largest in the country. In fact, its exchanges aggregate more than those of all the others together, averaging about $100,000,000 daily. The banks do not pay the balances to one another; but if a bank owes a balance of $100,000 to all the others, it pays this to the clearing house; and if another bank has a balance in its favor of $50,000, the clearing house pays it that sum. In this way, the amount coming into the clearing house must equal the amount to be paid out.

Exercises. 1. The exchanges in the New York Clearing House for one week were $623,405,190, and the actual balances paid were $36,951,619; what was the average of each per day, and the balances for the week were what per cent of the exchanges?

2. Since 1880 the highest average daily clearings for any one year at the New York Clearing House were $159,232,191 for 1881; the average daily balance paid in money was 3.5%; how much was this?

3. In the same period the lowest average daily balance paid in money was $4,247,069, which was 5.1% of the average daily clearings; to how much did the average daily clearings amount?

(*c*) **The postal money order** is substantially the same as a draft, except that instead of being drawn by a bank cashier on a bank in some large city it is drawn by one postmaster on another. These orders are always sold at a premium and cashed at par. The premium (price) varies from 3 cts. to 30 cts. depending on the amount, which may be from 1 ct to $100.

(*d*) **The express money order** is substantially like the postal money order.

(*e*) **The telegraphic money order.** Telegraph companies receive money at one office and telegraph (usually through the office of the manager of this part of the business) to another office to pay out an equal sum to the person named. A higher fee is charged than for drafts, but this method is employed when great promptness is necessary.

Exercises. 1. If you owed $25 in Syracuse, N. Y., which of the five methods named would you take to pay it? Why?

2. What would be the cost of a $500 draft at 0.1% premium? at par? at 0.1% discount?

3. If the fee for a money order over $75 and not exceeding $100 is 30 cts., what is the total cost of a money order for $87.50?

4. To send money by telegraphic order costs the double rate for a 10-word message and 1% of the sum sent; if the rate for a 10-word message is 40 cts., what would be the total cost of a telegraphic order for $87.50?

5. When a New York draft for $40,000 can be bought in St. Louis for $39,950, is exchange at a premium or a discount? What is the rate? How is the balance of trade?

6. Find the cost of each of the following drafts:

	Face.	Exchange.		Face.	Exchange.
(*a*)	$4350,	$\frac{1}{8}$% prem.	(*f*)	$1276.90,	$\frac{1}{4}$% prem.
(*b*)	$9275,	$\frac{1}{4}$% disct.	(*g*)	$2493.60,	par.
(*c*)	$2450,	$\frac{1}{2}$% "	(*h*)	$4275.75,	$\frac{1}{8}$% prem.
(*d*)	$7500,	$1.25 prem.	(*i*)	$10,023.60,	$1 disct.
(*e*)	$8556.75,	75 cts. disct.	(*j*)	$9870,	$1.50 prem.

7. The cost of a draft including the premium of 0.1% is \$4254.25; what is the face? (Business men usually merely subtract the premium from the cost, a process which, while not accurate, gives a sufficiently close result on sums below \$10,000. In this and the following exercises find both the correct result and the business approximation.)

8. Find both accurately and by the business approximation the face of each of the following drafts (see Ex. 7):

	Cost.	Exchange.		Cost.	Exchange.
(*a*)	\$5244.75,	\$1 disct.	(*d*)	\$5012.50,	$\frac{1}{4}$% prem.
(*b*)	\$1757.80,	$\frac{1}{8}$% "	(*e*)	\$4268.07,	25 cts. "
(*c*)	\$13,593.29,	$\frac{1}{8}$% prem.	(*f*)	\$14,518.13,	\$1.25 "

Foreign exchange is subject to the same general laws as domestic exchange, differing chiefly as to the currency and the manner of quoting the rate of exchange.

Thus, the *par of exchange* on London is 4.8665, that is, £1 in gold is worth \$4.8665 in gold. If exchange is selling at 4.90 it is above par, a draft for £1 costing \$4.90; while if it is selling at 4.84 it is below par. Exchange on London and other cities in Great Britain and Ireland is always quoted at so many dollars to the pound.

Exchange on Paris, and other cities in France and in countries like Belgium and Switzerland which use the French monetary system, is usually quoted at so many francs to the dollar, the quotation 5.14 meaning that \$1 will buy a draft for 5.14 francs. It is sometimes, and more conveniently, quoted at so many cents to the franc, the quotation 19.8 meaning that 19.8 cts. will buy a draft for 1 franc. The par of exchange is about $5.18\frac{1}{8}$, or 19.3.

Exchange on German cities is usually quoted at so many cents to 4 marks, the quotation 96 meaning that 96 cts. will buy a draft for 4 marks. It is sometimes, and more conveniently, quoted at so many cents to the mark, the quotation $23\frac{1}{2}$ meaning that $23\frac{1}{2}$ cts. will buy a draft for 1 mark. The par of exchange is about 95.2, or 23.8.

Foreign drafts are usually called *bills of exchange*, a bill at 30 days' sight being a draft due 30 days (or 30 days + 3 days of grace) after sight.

It is the custom with foreign bills, and occasionally with domestic drafts (sometimes called *inland bills*) between distant cities, to make out duplicates, as follows:

£50. New York.....[date]....., No. 147638.

At sight of this first of exchange (second of the same tenor and date unpaid) pay to the order of Brown Brothers & Co. fifty pounds sterling, value received, and charge the same to the account of

To Brown, Shipley & Co., London, England. } John Doe.

£50. New York.....[date]....., No. 147638.

At sight of this second of exchange (first of the same tenor and date unpaid) pay to the order of Brown Brothers & Co. fifty pounds sterling, value received, and charge the same to the account of

To Brown, Shipley & Co., London, England. } John Doe.

Such duplicate drafts form a *set of exchange.* Formerly three such drafts were made out; at present the leading dealers in foreign exchange draw their bills in duplicate; recently, in the case of express company drafts used by tourists, only a single bill is demanded, and the custom is extending.

Exercises. 1. What is the cost of a draft on London for £100, exchange 4.90? Is the balance of trade, judged by this quotation, against this country or against England?

2. What is the cost of a draft on Paris for 1000 francs, exchange 5.20? exchange 5.13? Which result is the greater? In which case is the balance of trade against this country? In which is it in favor of this country?

3. What is the cost of a draft on Paris for 1000 francs, exchange 19.8? exchange 19? Which result is the greater? In which case is the balance of trade against this country? In which case is it in favor of this country?

4. What is the cost of a draft on Leipzig for 500 marks, exchange 97? exchange 94? Consider the balance of trade in each case.

5. What is the cost of a draft on Hamburg for 200 marks, exchange $23\frac{1}{2}$? exchange 94?

6. Find the cost of each of the following drafts:

	Face.	Drawn on.	Rate of Exchange.
(*a*)	£700,	London,	4.84.
(*b*)	2750 francs,	Paris,	5.19.
(*c*)	6280 marks,	Frankfort,	$95\frac{1}{4}$.
(*d*)	£525 8 shillings,	Liverpool,	4.88.
(*e*)	1425 francs,	Paris,	$19\frac{1}{2}$.
(*f*)	800 marks,	Berlin,	$23\frac{5}{8}$.
(*g*)	£25 4 shillings,	London,	4.90.
(*h*)	750 francs,	Brussels,	5.18.
(*i*)	575 marks,	Leipzig,	24.
(*j*)	£50,	Glasgow,	4.87.
(*k*)	8760 marks,	Munich,	$95\frac{1}{2}$.

7. A New York merchant owes the following sums to foreign dealers; if he remits by draft, what is the face in each case?

	Amount Owed.	Draft Payable at.	Rate of Exchange.
(*a*)	$2435,	London,	4.87.
(*b*)	$1920,	Paris,	5.20.
(*c*)	$2400,	Leipzig,	96.
(*d*)	$1958,	Liverpool,	$4.89\frac{1}{2}$.
(*e*)	$81.62,	Paris,	$19\frac{1}{4}$.
(*f*)	$117.50,	Berlin,	$23\frac{1}{2}$.

8. An express company sells travelers' checks payable in various countries of Europe. It charges $\frac{1}{2}\%$ premium, and every $20 check allows the owner to draw £4 1*s.* 6*d.* in England, or 102.50 francs in France, or 82.50 marks in Germany. The company also has the use of the money until the checks are paid, an average of 2 mo., the use of the money being worth at the rate of 5% a year. On $500 of checks, paid in England, how much does the company make above the par of exchange?

9. In Ex. 8, suppose the checks paid in Germany.

10. In Ex. 8, suppose the checks paid in France.

11. The rates for foreign money orders, payable in the currency of the country to which they are sent, are: for sums not exceeding $10, 10 cts.; $10–$20, 20 cts.; $20–$30, 30 cts.; ····· $90–$100, $1. What would be the cost of the following money orders:

(*a*) $75 payable in London?
(*b*) $62.50 " Paris?

CHAPTER XVIII.

Government Revenues.

Certain revenues are necessary for the support of the governments of the United States, the various individual states, the counties, the cities, etc. The methods of obtaining these revenues are prescribed by law and vary for these different kinds of governments.

I. The United States Government.

The expenses of our general government are about a million dollars a day, and our income should be about the same or enough more to gradually reduce our indebtedness. Some of our sources of income and our principal expenditures are as follows, although all of the items vary from year to year :

Income.		Expenditures.	
Internal revenue		War dept.	$50,000,000
Spirits	$90,000,000	Navy "	$30,000,000
Tobacco	$30,000,000	Indians	$10,000,000
Fermented liquors	$32,000,000	Pensions	$135,000,000
Oleomargarine and other penalties	$2,000,000	Interest on debt	$30,000,000
Customs revenue,	$160,000,000 to $200,000,000	Diplomatic and consular service, and miscellaneous,	$100,000,000
Total, including the above and the income from public lands, etc.,	$325,000,000 to $400,000,000	Total,	$325,000,000 to $400,000,000

The *customs revenue* (*tariff, duty*) is collected at *custom houses* situated at *ports of entry* established by law.

Merchandise brought into the country (1) is on the *free list* (*i.e.*, it is not subject to duty), or (2) is subject to *ad valorem duty* (a certain per cent on the value at the place of purchase), or (3) is subject to *specific duty* (a certain amount by number, measure, etc.), or (4) is subject to *both ad valorem and specific duty.*

E.g., by the tariff of 1883 apples were on the *free list;* by the tariff of 1890 they paid a *specific duty* of 25 cts. a bushel; by the tariff of 1894 they paid an *ad valorem* duty of 20%. By the tariff of 1890 oriental rugs paid a *specific duty* of 60 cts. per sq. yd., and an *ad valorem* duty of 40%.

Ad valorem duty is, if honestly collected, the more fair; but on account of undervaluation by the importer there is much more chance for fraud in the collection.

Exercises. 1. Taking the total revenue of our general government for one year as $326,926,200, and our internal revenue as $146,762,865, what per cent of our income was of this class?

2. In one year the internal revenue was $143,421,672, including $79,862,627 from spirits, $29,707,908 from tobacco, and $31,640,618 from fermented liquors; what per cent was derived from these three classes?

3. The revenue of the post office department for a certain year was $76,983,128, and the expenditures were $86,790,172; the excess was what per cent of the revenue?

4. Mathematical instruments pay a duty of 35%; what is the invoice price of an instrument which pays a duty of $6.30?

5. The duty on cheese is 4 cts. per lb.; how much does a city which consumes 40,000 lbs. of French cheese a year pay to the government for this privilege?

6. The duty on aniline dyes being 25%, what is the valuation at the custom house on a package of dyes which pays $59.38?

7. The duty on fine blankets being 35%, what is the invoice price of a shipment of blankets which cost the importer $786.73, including the duty and $12.50 freight?

8. Rubber coats pay a duty of 40%; how much is the duty on 100 doz. invoiced at £1 1s. a dozen, reckoning the pound at 4.86\frac{2}{3}$?

9. Ready-made woolen clothing pays a duty of 50%; how much less would a $20 suit cost if it were on the free list, not considering the freight and profit?

10. Cutlery valued from $1.50 to $3 a dozen pays a duty of 75 cts. a dozen and 25% ad valorem; what is the duty on 100 doz. Sheffield knives invoiced at $2.25 a dozen?

11. English books pay a duty of 25%; how much less would you have to pay for an English book which costs you $8, including the duty and 50 cts. postage, if it were not for this tariff?

II. State and Local Taxes.

The method of collecting taxes varies in different states, but in general it may be said that a valuation is placed upon the property of corporations, of land owners, and of persons possessing any considerable amount of personal property. Upon this *assessed valuation* a certain *rate of taxation* is fixed.

The expression *rate of taxation* is usually applied to the number of mills of tax on each dollar of valuation.

Thus, if the rate is $5\frac{1}{2}$ mills, the tax is $5\frac{1}{2}$ mills on each dollar.

The rate of taxation is, therefore, found by dividing the amount to be raised by the *number* of dollars of valuation.

E.g., if a village has to raise $12,575, and if the valuation is $2,465,685, the rate of taxation is $\frac{\$12,575}{2,465,685} = \$0.0051$ on $1.

In addition to the tax already mentioned, male citizens over 21 years of age are frequently required to pay a *poll* (*i.e.*, head) *tax*.

If taxes are not paid when due a fine is usually imposed in the form of a certain per cent of increase of the tax.

E.g., if a man's taxes are $12, and he does not pay them when due, the law may require him to pay 5% additional, thus making his tax $12.60.

Tax collectors usually prepare a table similar to that given below. For this table, the rate of $5\frac{1}{2}$ mills on $1 has been taken.

TAX TABLE. RATE $5\frac{1}{2}$ MILLS ON $1.

	0	1	2	3	4	5	6	7	8	9
0	0000	0055	0110	0165	0220	0275	0330	0385	0440	0495
1	0550	0605	0660	0715	0770	0825	0880	0935	0990	1045
2	1100	1155	1210	1265	1320	1375	1430	1485	1540	1595
3	1650	1705	1760	1815	1870	1925	1980	2035	2090	2145
4	2200	2255	2310	2365	2420	2475	2530	2585	2640	2695
5	2750	2805	2860	2915	2970	3025	3080	3135	3190	3245
6	3300	3355	3410	3465	3520	3575	3630	3685	3740	3795
7	3850	3905	3960	4015	4070	4125	4180	4235	4290	4345
8	4400	4455	4510	4565	4620	4675	4730	4785	4840	4895
9	4950	5005	5060	5115	5170	5225	5280	5335	5390	5445

The column at the left gives the first figure of the number of dollars of valuation, and the row at the top the second figure. A decimal point is understood before each of the other numbers.

E.g., the tax on $10 is $0.055,
" $87 " $0.4785,
" $5900 " $32.45.

To find the tax on $9805, the collector's commission being 1%, the actual computation of a collector would be as follows:

Tax on $9800	=	$53.90
" $5	=	.03
		$53.93
Commission 1%	=	.54
		$54.47

Exercises. 1. From the preceding table find the tax at $5\frac{1}{2}$ mills on $1 on each of the following valuations, the collector's commission being 1%: (*a*) $1750, (*b*) $2500, (*c*) $5475, (*d*) $17,645, (*e*) $18,750, (*f*) $9250, (*g*) $7625.

2. Prepare the first two rows of a tax table (opposite 0 and 1) at the rate of $8\frac{1}{2}$ mills on $1.

3. From Ex. 2, compute the tax on each of the following valuations at $8\frac{1}{2}$ mills on \$1, collector's commission being 1%: (*a*) \$1200, (*b*) \$19,150, (*c*) \$15,175, (*d*) \$1750, (*e*) \$17,150, (*f*) \$1825, (*g*) \$825, (*h*) \$500.

4. Taxes are levied in a certain village as follows: for streets \$2000, for fire apparatus, etc. \$1500, for school purposes \$6000, for salaries and office rent \$3400, for repair of bridge \$500, for general purposes \$500; the total valuation of property is \$1,950,000; what is the rate of taxation?

5. In Ex. 4, what would be the taxes of a man whose property is valued by the assessors at \$2500?

6. The rate of taxation being $5\frac{1}{2}$ mills on \$1, what are the taxes on property valued by the assessors at \$9500, collector's commission 1%? Suppose the owner does not pay promptly and is fined 5%, what is his tax? (Use the table.)

7. Find the tax on

(*a*) \$18,500 at 4 mills on a dollar,
(*b*) \$6000 " 3.8 "
(*c*) \$3500 " 8.4 "
(*d*) \$21,400 " 7.2 "
(*e*) \$5500 " 5 "
(*f*) \$6750 " 4.8 "
(*g*) \$1800 " $5\frac{1}{2}$ "

8. The assessed valuation of a district being \$950,725, what is the rate of taxation necessary to raise \$8000?

9. To raise \$2000, a tax of $1\frac{1}{4}$ mills on a dollar was levied; what was the assessed valuation?

10. What would be the various taxes levied on a man whose property is valued by the assessors at \$12,800, if the rates were as follows: state tax $1\frac{1}{2}$ mills, county 2 mills, town 0.8 mill, school 1.4 mills?

11. At 7 mills on a dollar, how much is the tax of a man who owns a farm of 250 acres, worth \$70 an acre, but assessed for only $\frac{3}{5}$ of its value?

12. The rate of taxation in a certain town is 5 mills on a dollar, and the amount to be raised is \$4783.87; what is the assessed valuation?

13. At $6\frac{1}{2}$ mills on a dollar, how much is the tax of a man who owns a farm of 300 acres assessed at \$10 an acre, and who is assessed on \$2000 of personal property, and who pays a poll tax of \$1?

CHAPTER XIX.

Commission and Brokerage.

Produce bought in quantities or sent to cities for sale is usually bought or sold through a *commission merchant* or a *broker*.

A *commission merchant* usually has the goods consigned to him and sells them in his own name, remitting the *net proceeds* (the sum realized less the *commission*) to the consignor. If he is buying for a customer, he charges the sum paid plus his commission.

A *broker* does not receive the goods, but sells them for the consignor in advance or buys them for his customer, and they are shipped directly to the buyer. His commissions, called *brokerage*, are therefore less than those of the commission merchant.

Commission and brokerage are reckoned as a certain per cent of the amount paid in buying or realized in selling, but more often as a certain amount for a given transaction.

Thus, it is more common to pay 50 cts. a ton for selling hay than to pay a commission of 4% or 5%. Stocks are bought and sold on a brokerage of 12½ cts. for each share, as explained in Chap. XX.

There are numerous other cases involving commission and brokerage, as the buying and selling of securities. Some of these are mentioned in the exercises.

Since no new principles are involved, illustrative problems are unnecessary.

Exercises. 1. What is the commission for buying a carload of 400 bu. of grain at $\frac{1}{2}$ of a cent a bushel? for selling 4 carloads of hay at \$5 a car? for selling 500 bu. of beans at 95 cts. a bushel, commission 5%?

2. How much does a broker receive for selling 1200 bales of cotton, brokerage 25 cts. a bale? 500 bbls. of rye flour at \$2.95, brokerage $2\frac{1}{2}$%? 10,000 bu. wheat, brokerage $\frac{1}{8}$ (of a cent a bushel)?

3. A commercial traveler sells goods at a commission of 3%; to how much must his sales amount that he may have an income of \$4500 a year?

4. A commission merchant receives 100 boxes of Mexican oranges which he sells at \$3.50 a box and remits \$323.80 net proceeds; what is the rate of his commission?

5. What are the net proceeds of a sale of 8750 lbs. of leather at $25\frac{1}{2}$ cts. a pound, commission $2\frac{1}{2}$%?

6. A speculator buys 1000 bbls. of May pork (*i.e.*, to be delivered the following May) at \$7.82$\frac{1}{2}$, and sells it at \$7.90; he pays a brokerage of $2\frac{1}{2}$ cts. (on each barrel) for buying and the same for selling; does he gain or lose, and how much?

7. A speculator buys 10,000 bu. of May wheat at $83\frac{5}{8}$ (cts. a bushel) and sells it at 82; the brokerage is $\frac{1}{8}$ (of a cent a bushel) for buying and the same for selling; how much does he lose?

8. An auctioneer offers his services at \$8 a day or 2% of amount sold; a merchant accepts the latter offer and the stock is disposed of in 4 da., realizing \$1875.50; how much less would he have paid if he had taken the first offer?

9. A collector has a \$500 note placed in his hands with power to compromise; he accepts 75 cts. on a dollar and charges 5% of the sum collected, and 25 cts. for a draft; what are the net proceeds?

10. A broker buys flour for a customer at \$3.30 a barrel, charging 2%; the bill, including commissions, is \$4039.20; how many barrels are bought?

11. A dealer buys 1000 doz. eggs at an average price of 16 cts. a dozen and sends them to a commission merchant who sells them at an advance of 4 cts. a dozen, charging 10% commission; the express was \$7.50; did the dealer gain or lose, and how much?

12. At 5%, what is the brokerage for selling 1000 bu. of potatoes at 38 cts. a bushel?

13. A commission merchant sells 275 bu. of onions at 60 cts. a bushel, and remits the proceeds after deducting his commission of $7\frac{1}{2}$%; what is the amount remitted?

14. A commission merchant remits \$266 as the proceeds of a sale of 200 bbls. of apples, his commission being 5%; at what price per barrel did he sell them?

15. A man sends a carload of 13 tons of hay to Boston where it sells for \$14 a ton, and receives \$175.50 after paying his broker; how much was the brokerage a ton?

16. In Ex. 15, if the hay cost the man \$8.50 a ton, and the freight cost 21 cts. a 100 (lbs.), did he make or lose by the transaction, and how much?

17. A commission merchant sells 4000 heads of cabbage at \$3.50 a hundred, and remits \$126; what was his rate of commission?

18. 400 bu. of beans at 62 lbs. to the bushel are shipped to Boston, the freight being 28 cts. a 100 (lbs.); the beans cost the shipper 70 cts. a bushel and were sold through a broker at 95 cts., brokerage 5%; how much did the shipper gain?

19. A commission merchant sold 600 lbs. of butter at 24 cts., 480 doz. eggs at 20 cts., 1200 lbs. poultry at 7 cts.; what are the net proceeds after deducting \$14 for freight and cartage and $2\frac{1}{2}\%$ commission?

20. A broker remits \$1706 after deducting $2\frac{1}{2}\%$ for brokerage and 25 cts. for the draft; how much was his brokerage?

21. A salesman received \$6782.88 in one year, this representing his commissions at $1\frac{1}{4}\%$; find the amount of his sales.

22. A lawyer having a debt of \$3250 to collect, compromises for $97\frac{1}{2}$ cts. on a dollar; his commissions are $2\frac{1}{4}\%$; how much does he remit to his client?

23. A lawyer collects a debt for a client, takes $3\frac{1}{4}\%$ for his pay, and remits the balance, \$1935; what was the debt and the fee?

24. An agent buys goods on commission at $2\frac{1}{2}\%$, and pays \$40 for freight; the whole amount was \$1628.73; what was the sum expended for goods?

25. A real estate agent sold some western land for a man and, after retaining \$23.40 as his commission, remitted \$2116.60; what rate of commission did he charge?

26. An agent sells some property for s dollars on a commission of $r\%$; what are the net proceeds?

27. The net proceeds from the sale of some property is p dollars, and the rate of commission is $r\%$; at what price was it sold?

28. An agent sells some property for s dollars and remits p dollars as the net proceeds; what was the rate of commission?

CHAPTER XX.

Stocks and Bonds.

When a number of persons wish to engage in business the law allows them to form a corporation usually known as a *stock company* with a certain *capital stock*, each person owning a certain number of *shares* of that stock, each share being allowed one vote at the meetings for the election of directors. The business of these companies is managed by officers, usually elected by the directors.

If the company makes more than its expenses, part or all of the surplus is divided among the stockholders in the form of *dividends*.

If a stock is paying a good rate of dividend, that is, a higher rate than can be received from ordinary investments, a $100 share will cost more than $100, and the stock is said to be *above par*. If it is paying about the same rate that ordinary investments bring, a $100 share may be bought for $100, and the stock is said to be *at par*. If it is paying low dividends, or none, it will be *below par*.

The dividends are expressed either as a certain per cent of the *par value* or as a certain number of dollars per share. *E.g.*, a 5% stock is one which is paying 5% on the par value; and if stock is paying a dividend of $3 it pays $3 on each share. If the par value of one share is $100, then a stock paying $3 a share is the same as a 3% stock. But if, as is often the case with mining stocks, the par value is $25, a dividend of $3 a share is at the rate of 12%.

Sometimes a company issues two kinds of stock, *preferred*, which is entitled to the dividends to a certain amount (*e.g.*, to 5% of the par value), and the *common*, which is entitled to part or all of the balance.

On Jan. 1, 1897, the total capital stock of all steam railways in North America was $5,008,352,237, of which $3,986,753,937 was common stock and $1,021,598,300 was preferred.

When a company needs more money than has been paid in by the stockholders it often borrows money and issues *bonds* payable at a certain time and bearing a certain rate of interest.

These bonds are usually secured by a mortgage on the property of the company, taken in the name of trustees for the bondholders.

Similarly, when a national, state, county, or city government wishes to borrow money it issues bonds, but without mortgages.

Bonds either have *coupons* annexed, which are cut off as interest becomes due and are collected for the owner by the bank where he keeps his account, or are *registered*, that is, bear the name and address of the owner, the interest being sent when due.

Bonds are spoken of as "4's reg.," "5's coup.," etc., meaning that they draw 4% of their par value and are registered, or draw 5% of their par value and have coupons annexed.

Exercises. 1. Which would you prefer to own, common stock or preferred stock? Why? Suppose the preferred stock paid 5% and the common 7%? the common 3%?

2. Suppose the capital stock of a company is $100,000, half being preferred and half common, the former being entitled to 5% and the latter to the balance; suppose $6000 to be distributed in dividends, what rate of dividend would be received by the common stock? If the dividends remain the same from year to year, which kind of stock would you prefer to have at the same price?

3. A company having $100,000 capital, of which $30,000 is preferred stock entitled to 5%, the balance going to the common stock, has $6000 available for dividends; what is the rate of dividend of the common stock?

4. Which would you prefer to own, $1000 of stock in a certain railway, or one of its $1000 bonds? Suppose it was a 5% bond, while the stock paid 7%? 5%? 4%?

5. Which would you prefer to own, a coupon bond or a registered bond? Why? Which is the safer against loss by theft? Which is the more easily transferred in case you wish to sell?

6. A certain railway stock is paying 9% dividends annually, and another is paying 2%; are they above or below par? Why?

7. United States 4% bonds are sold at 117, that is, a $100 bond costs $117, while Atchison railway 4% bonds are sold at 80; what is the reason for this difference in price?

8. In 1896 $68,981,244 of dividends was paid on $3,986,753,937 of common stock in the North American railway companies, and $16,533,019 on $1,021,598,300 of preferred stock; what was the average dividend in each case?

Purchasing stocks and bonds. Since one usually does not know who has stock for sale he applies (directly or through a bank) to a *stock broker* who belongs to some *stock exchange*, where stocks and bonds are bought and sold. The leading stock exchange of the United States is in New York.

The broker charges *brokerage*, usually $\frac{1}{8}$% *of the par value.* This is charged for buying and also for selling.

A newspaper quotation of 122 means that $100 of stock, which we shall always take as representing the par value of one share, as is usually the case with railway stocks, is selling for $122. But the seller would receive only $122 − $$\frac{1}{8}$, or $$121\frac{7}{8}$, for each share, because he must pay his broker; and the buyer must pay $122 + $$\frac{1}{8}$, or $$122\frac{1}{8}$, for each share, because he too must pay his broker.

In stock quotations, fractions are expressed in eighths, quarters, or halves. *E.g.*, stock is often quoted at $97\frac{5}{8}$, but never at $62\frac{2}{3}$.

Fractions of a share are not usually sold; if a person has $1000 to invest in Canada Southern Railway stock, quoted at $48\frac{7}{8}$, he would pay $49 a share, and purchase 20 shares and have $20 left.

The purchaser receives a *certificate of stock*, signed by the proper officers of the company, stating that he owns so many shares. When he sells his stock he sends this certificate, properly indorsed, to his broker; it is delivered to the company and another certificate is made out for the new purchaser.

Newspaper quotations of the prices of stocks and bonds are given in the daily papers and form the best basis for a series of problems. The brokerage must be considered in each case. In the absence of a daily paper the following quotations may be used, and on them are based the problems on pp. 175, 176.

STOCKS.		BONDS.	
Atchison	$16\frac{1}{2}$	U. S. 4's reg.	$116\frac{3}{4}$
" prefd.	$25\frac{3}{8}$	U. S. 4's coup.	117
C. B. & Q.	$79\frac{5}{8}$	U. S. 2's	95
C. & N. W.	$104\frac{1}{4}$	Atchison 4's	80
Canada South.	51	Balt. & Ohio 5's	99
N. J. Central	$107\frac{1}{4}$	Erie 7's	137
Canadian Pacif.	57	North. Pac. 6's	$114\frac{1}{4}$
D. L. & W.	$159\frac{7}{8}$	Wabash 5's	$107\frac{1}{2}$
Lake Shore	$148\frac{1}{2}$	N. J. Central 5's	$118\frac{1}{4}$
N. Y. Central	97	Ill. Central $4\frac{1}{2}$'s	$110\frac{1}{4}$
Pullman Car Co.	158	C. B. & Q. 5's	100

Illustrative problems.

1. Suppose a man buys 10 shares of Atchison as quoted above and sells them 6 mo. later when quoted at 18, having received no dividends; does he gain or lose, and how much, money being worth at the rate of 4% a year to him?

1. He buys for $16\frac{1}{2} + \frac{1}{8}$, and sells for $18 - \frac{1}{8}$,
   $\therefore$ he gains $1\frac{1}{4}$, that is, \$1.25 on a share.
2. $\therefore$ " $10 \cdot \$1.25$, or \$12.50.
3. But he loses $\frac{1}{2}$ of 4% of $10 \cdot (\$16\frac{1}{2} + \$\frac{1}{8})$, or \$3.33, interest.
4. $\therefore$ his net gain is $\$12.50 - \$3.33$, or \$9.17.

2. Suppose a man buys 50 shares of C. & N. W. as quoted above and sells them 6 mo. later when quoted at $102\frac{1}{2}$, meanwhile receiving a 3% dividend; does he gain or lose, and how much, money being worth at the rate of 5% a year to him?

1. He loses $(\$104\frac{1}{4} + \$\frac{1}{8}) - (\$102\frac{1}{2} - \$\frac{1}{8})$, or \$2, on a share.
2. $\therefore$ " $50 \cdot \$2 = \$100$.
3. He also loses $2\frac{1}{2}\%$ of $50 \cdot (\$104\frac{1}{4} + \$\frac{1}{8}) = \$130.47$, interest.
4. $\therefore$ his total loss is $\$100 + \$130.47 = \$230.47$.
5. He gains 3% of $50 \cdot \$100 = \$150$.
6. $\therefore$ his net loss is $\$230.47 - \$150 = \$80.47$.

3. Not considering the length of time the bond runs, what rate of income does a purchaser receive from investing in U. S. 4's reg. as quoted on p. 174?

1. He receives \$4 on every ($\$116\frac{3}{4} + \$\frac{1}{8}$) invested.
2. Let $r\%$ stand for the rate.
3. $\therefore r\%$ of $\$116\frac{7}{8} = \$4$.
4. $\therefore r\% \qquad = \dfrac{\$4}{\$116\frac{7}{8}} = 3.42\%$.

Exercises. Unless otherwise directed, use the quotations given on p. 174, remembering the brokerage in each case. In finding the rate of income on bonds the time of maturity is not considered in these exercises.

1. What will 20 shares of Pullman Car Co. stock cost?

2. Also 125 shares of N. Y. Central?

3. Also 75 shares of Lake Shore?

4. What sum will be received from the sale of 10 shares of C. B. & Q.?

5. Also from the sale of 40 shares of D. L. & W.?

6. Suppose a man buys 100 shares of N. J. Central as quoted and sells it when quoted at $115\frac{3}{8}$, what is the gain, not considering dividends or interest?

7. Solve Ex. 6, supposing the stock had paid a 2% dividend meanwhile, and that 8 mo. had elapsed and that money was worth at the rate of 6% a year to the investor.

8. Suppose a man sells 100 shares of Canada Southern as quoted, this stock paying $2\frac{1}{2}\%$ dividends annually, and invests the proceeds in 31 shares of D. L. & W. which pays 9% dividends annually, putting the balance in a savings bank where it draws 4%; find the alteration in income.

9. Suppose a man sells 50 shares of C. B. & Q., which pays 4% dividends, and invests the proceeds in Lake Shore, which pays 8%, buying as many shares as possible, and placing the balance in a savings bank where it draws 4%; find the alteration in income.

10. Suppose a man has \$2500 to invest; what is the greatest number of shares of Pullman Car Co. that he can buy, and how much will he have left?

11. Which investment pays the better, a 5% bond and mortgage or Erie 7's as quoted, the interest being paid promptly?

12. Also Erie 7's or North. Pac. 6's?

13. Also U. S. 2's or U. S. 4's coup.?

14. Also C. B. & Q. 5's or North. Pac. 6's ?

15. A man's income in Erie 7's is $245; how much has he invested, at par value? How much did the bonds cost him, as quoted?

16. A man's income in D. L. & W. stock, while it pays 9% annually, is 6% on the sum invested; what was the quotation when he made the investment?

17. A man's income is $5\frac{5}{7}\%$ on the sum invested in C. & N. W. which he purchased when quoted at $104\frac{7}{8}$; find the rate of dividends.

18. A broker bought on his own account 50 shares of Atchison prefd. as quoted and sold it at $27\frac{1}{8}$; how much did he gain?

19. Tamarack Mining Co. stock pays a semi-annual dividend of 3%; how much will the holder of 50 $100-shares receive?

20. A bank with a capital of $150,000 declares a quarterly dividend of 2%; what is the total amount of this dividend and how much will the owner of $1200 of stock receive?

21. To raise more money a company sometimes assesses its stockholders. If a certain mining company levies an assessment of 10%, how much must be paid by the holder of 50 $100-shares?

22. How much must be invested in Wabash 5's as quoted to bring an annual income of $1000?

23. The common stock of a certain railway company is $20,000,000, and the preferred stock (which in this case is entitled to 6% annually) is $4,000,000. The company declares a semi-annual dividend, paying the usual amount to the preferred stockholders and $2\frac{1}{2}\%$ to the others. How much money was distributed in dividends?

24. How much must be invested in Ill. Central $4\frac{1}{2}$'s as quoted to bring an annual income of $1350?

25. How much income will be derived from an investment of $991.25 in Balt. & Ohio 5's as quoted?

26. A certain stock is quoted at 260; a broker is instructed to buy a certain number of shares at this price; his bill including brokerage is $2081; how many shares did he buy?

27. The average rate of dividends paid to stockholders in national banks in 1872 was 10.19%, the dividends amounting to $46,687,115; in 1895, 6.96%, amounting to $45,969,663; what was the total capital for each of these years?

28. In Ex. 27, find the rate of increase of capitals and the rate of decrease of dividends.

29. Of the bonds quoted on p. 174, which yields the highest rate of income on the investment?

30. Of the same bonds, which yields the lowest rate of income?

CHAPTER XXI.

Insurance.

For the majority of citizens insurance business is confined to three general lines, the practical problem being substantially the same in all cases. The three lines are

1. Fire insurance,
2. Life insurance,
3. Accident insurance,

and the practical problem is, *Given the face of the policy and the rate to find the premium.*

Less common are such special forms as tornado, plate glass, and steam-boiler insurance, insurance against loss by theft, marine insurance, etc. The technical features of insurance are so constantly changing that it is inexpedient to enter into the subject with any detail.

The premium is computed either as a certain per cent of the face of the policy, or, what is analogous to it, as a certain sum on each \$100 of insurance. The latter is the usual form. Both this certain per cent and this certain sum go by the name *rate of insurance.*

E.g., the rate for insuring the life of a man 30 yrs. old in a certain company, the policy to mature at death, is \$22.85 annually on \$1000, although it might be stated as 2.285%.

The rate for insuring a business block against fire for 1 yr. (the usual time for insuring places of business) may be \$1.10 on \$100.

The rate for insuring a house against fire for 3 yrs. (the usual time for insuring dwelling houses) may be \$0.95 (for the 3 yrs.) on \$100.

Exercises. 1. What are the premiums for insuring business property against loss by fire for 1 yr. for the following amounts at the specified rates?

(*a*) $2000 at $0.90 per $100,	contents for	$5000 at $0.95 per $100.
(*b*) $3500 " $1.10 "	"	$10,000 " $1.25 "
(*c*) $8000 " $1.35 "	"	$50,000 " $1.40 "
(*d*) $7500 " $1.20 "	"	$35,000 " $1.30 "

2. What are the premiums for insuring dwelling property against loss by fire for 3 yrs. for the following amounts at the specified rates?

(*a*) $1000 at $0.90 per $100,	contents for	$1500 at same rate.
(*b*) $7000 " $1.10 "	"	$5000 "
(*c*) $6500 " $0.95 "	"	$4000 "
(*d*) $4000 " $1.05 "	"	$3750 "

3. What are the premiums for insuring manufacturing establishments against loss by fire for 1 yr. for the following amounts at the specified rates?

(*a*) $10,000 at $2.25 per $100.
(*b*) $50,000 " $1.75 "
(*c*) $25,000 " $1.95 "
(*d*) $15,000 " $2.10 "

4. What is the annual premium for insuring against loss by fire a business block for $8000 at $1.10 per $100, its ground and first floor contents for $10,000 at $1.20 per $100, its other contents for $8000 at $1.35 per $100, and two plate glass windows against damage from other causes than fire at $2.70 per window?

5. What is the annual premium for insuring a leaded glass window in a church for $1250 at 2%?

6. How much would be the annual premiums paid by a man 30 yrs. old for $5000 of life insurance at $22.85 per $1000? on the 10-payment plan (of paying only ten times, the policy maturing at death) at $54.65 per $1000? on the 25-payment plan at $28.46 per $1000? on the single-payment plan at $428.14 per $1000?

7. As in Ex. 6, on a 10-year endowment policy (one in which ten payments are made, the policy then maturing, or maturing at death if before 10 yrs.) for $5000 at $106.75 per $1000? on a 25-year endowment policy for $5000 at $38.85 per $1000?

8. What is the premium on a $1000 tornado insurance policy for 1 yr. at 20 cts. per $100? for 3 yrs. at 50 cts. per $100? for 5 yrs. at 80 cts. per $100?

CHAPTER XXII.

Miscellaneous Exercises.

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1. Multiply 1854.362 by 0.000087931, correct to 0.000001.

2. Multiply 162.5473 by 8726.47231, correct to 0.0001.

3. The distance of the moon from the earth is 59.97 times the earth's radius; if this radius is 3962.824 mi., find the distance to the moon, correct to 1 mi.

4. Divide 634.7538292 by 0.0657391, correct to 0.001.

5. Divide 15.63214725 by 0.0057123, correct to 0.001.

6. How many days, hours, minutes, and seconds in a year of 365.24226 da.?

7. How often does the heart beat in a life of 75 yrs. of 365 da. each, supposing that the number of beats is 140 per min. during the first 3 yrs. of life, 120 for the next 3, 100 for the next 6, 90 for the next 10, 75 for the next 28, 70 for the next 20, and 80 for the last 5?

8. Knowing that

$$1,040,318,228,677 = 2,870,564 \times 362,407 + 5,741,129,$$

state the quotient and the remainder from dividing 1,040,318,228,677 by 2,870,564; also by 362,407.

9. Supposing a person can count one hundred in 30 secs., and that after counting incessantly for 30 yrs. he dies, and his son goes on counting for 30 yrs. and then dies, and so on; how many generations must elapse before one trillion is counted?

10. A person loses $\frac{1}{10}$ of his fortune and then $\frac{1}{15}$ of the remainder; would the result have been the same if he had first lost $\frac{1}{15}$ and then $\frac{1}{10}$ of the remainder? Generalize for $\frac{1}{a}$ and $\frac{1}{b}$.

11. How much is the fraction $\frac{13}{8}$ increased or diminished when 5 is added to each term?

12. In any year show that the same days of the month in March and November fall on the same day of the week.

13. Reduce to simplest form $\dfrac{11\frac{5}{7} - 7\frac{5}{11}}{3\frac{1}{2} + 5\frac{5}{22}}$.

14. Reduce the fraction $\dfrac{\frac{5}{6} + \frac{1}{2}\left(\frac{2}{3} - \frac{1}{2}\right) - \frac{6}{7}\left(\frac{4}{5} + \frac{1}{9}\right)}{\frac{3}{4}\left(\frac{7}{8} - \frac{1}{2}\right) - \frac{1}{3}\left(\frac{1}{7} - \frac{1}{10}\right)}$ to its simplest form.

15. Simplify the expression $\frac{1}{13}$ of $\dfrac{1}{1 + \dfrac{\frac{1}{3}}{3 + \frac{1}{4}}}$.

16. Simplify the expression $\dfrac{\frac{1}{2} + \frac{1}{3} + \frac{1}{4}}{\dfrac{1}{2\frac{1}{2}} + \dfrac{1}{3\frac{1}{2}} + \dfrac{1}{4\frac{1}{2}}}$.

17. Divide $3\frac{1}{4} - \frac{5}{6}$ of $\frac{4}{15}$ by $21\frac{1}{5} + \frac{3}{10} + 4\frac{1}{3} \times 5$.

18. The first of a series of cog-wheels, working into one another in a straight line, has $7n$ teeth; the second has $6n$, the third $5n$, and the number in the fourth is to that in the third as 2 to 3. If the wheels are set in motion, how many revolutions must each make before they are simultaneously in their original positions?

19. Show that with a 1-ct. piece, two 2-ct. pieces, a 5-ct. piece, four dimes, a half-dollar, and nine silver dollars one can pay any sum less than or equal to \$10.

20. The moon revolves about the earth in 27 da. 7 hrs. 43 mins. 11.5 secs.; what is the average angle passed over in a day?

21. The length of an arc of $97° 21' 47.2''$ is 23 in.; find, correct to $0.1''$, the arc of the same circle 1 in. long.

22. What fraction of the circumference is an arc of $27° 17' 30''$? (Answer correct to 0.0001.)

23. Prove that the sum of a common fraction and its reciprocal is greater than 2. Is there any exception?

24. Prove that if the same number is added to both terms of a fraction the new fraction is nearer unity than the old.

25. Prove that, of three consecutive numbers, the difference between the squares of the first and third is four times the second.

26. Prove that the difference between two numbers composed of the same digits, as 937 and 793, is a multiple of 9.

27. Show that the integral part of a quotient is not changed by adding to the dividend a number less than the difference between the divisor and the remainder.

28. Given the sum and the difference of two numbers, show how to find each. Prove your statement.

29. Is the product of two square numbers always a square?

30. Prove that no number ending in 5 and not in 25 can be square.

31. Prove that a fraction whose terms are composed of the same number of digits is not altered in value by repeating the same number of times the figures of both terms. *E.g.*, $\frac{27}{34} = \frac{2727}{3434} = \frac{272727}{343434} \cdots\cdots$

32. What is the common fraction which, reduced to a decimal fraction, equals $0.4275275 \cdots\cdots$?

33. Prove that an integer cannot have for a square root a fractional number.

34. Prove that any odd square number diminished by 1 is a multiple of 8.

35. State the test of divisibility of a number by 33.

36. Show that if two numbers are prime to one another, any powers to which they may be raised are also prime to one another.

37. Extract the 32d root of 429,497,296.

38. Show that a factor of each of two numbers is also a factor of their greatest common divisor.

39. If your school building is heated by a furnace, compare the area of a cross section of the cold-air pipe with the sum of the areas of cross sections of the hot-air pipes, and determine the ratio.

40. What is the ratio of an arc of $321° 22'$ to one of $37° 21' 1''$?

41. Divide the arc of $88° 27' 33''$ into three parts proportional to the numbers 3.2, 5.6, 8.5.

42. Divide the length of 28.75 in. into three parts proportional to the numbers $\frac{3}{4}$, $\frac{5}{6}$, $\frac{9}{10}$.

43. How many pounds each of nickel and lead must be added to an alloy weighing 10 lbs. and consisting of 11 parts (by weight) nickel, 7 parts tin, and 5 parts lead, so that the new alloy shall consist of 19 parts nickel, 41 parts tin, and 17 parts lead?

44. It takes a letter 43 da. to go from New York to Siam, a distance of 12,990 mi., and 34 da. to go to Adelaide, Australia, a distance of 12,845 mi. What is the ratio of the average rate on the latter route to that on the former?

45. The area of Lake Superior is 32,000 sq. mi. and it drains an area of 85,000 sq. mi.; the area of Lake Erie is 10,000 sq. mi. and its drainage is 39,680 sq. mi. Are the areas proportional to the drainage? If not, what would be the drainage of Lake Superior to make them so?

46. What force can a man weighing 165 lbs. exert on a stone by pressing on a horizontal crowbar 6 ft. long, propped at a distance of 5 in. from the point of contact with the stone, not considering the weight of the bar?

47. A uniform rod 2 ft. long weighs 1 lb.; what weight must be hung at one end in order that the rod may balance on a point 3 in. from that end?

48. Two men carry a weight of 20 lbs. on a pole, one end being held by each; the weight is 2 ft. from one end and 5 ft. from the other; how many pounds does each support?

49. In a pair of nut crackers the nut is placed 1 in. from the hinge, and the hand is applied at a distance of 6 in. from the hinge; if the nut requires a force of 22.5 lbs. to break it, how much pressure must be exerted by the hand?

50. Three persons are associated in a common enterprise, the first having invested \$4000, the second \$7000, and the third \$9000. At the end of a year their gains amount to \$7340, out of which they pay the first \$2000 for managing the business and divide the balance among the three in proportion to their investments. How much did each receive?

51. A country is 600 mi. long and 320 mi. wide; find the dimensions of the paper on which a map of the country might be drawn, the scale being $\frac{1}{8}$ in. to the mile.

52. The average number of deaths in the world each minute is estimated at 67, and the average number of births at 70; how many of each in a year of 365 da.?

53. By what fractional part of an inch should the highest mountain in Alaska, 19,500 ft., be represented on a globe 16 in. in diameter, the earth's radius being taken as 4000 mi.?

54. In a certain enterprise in which three persons are engaged, A puts in \$3500 for 25 mo., B \$2400 for 15 mo., C \$4500 for 12 mo.; they gain \$5000; what is the share of each?

55. A, B, and C rent a pasture for 6 mo. for \$100; A puts in 25 cattle for the whole time, B 30 for $4\frac{1}{2}$ mo., C 45 for $3\frac{1}{3}$ mo.; find the rent paid by each.

56. The streets of a certain city have an area of 8 km^2. In a certain storm the average depth of snow was 25 cm. Assuming 12 cm^3 of snow to produce 1 cm^3 of water, find the volume of water produced by this snow, and the weight in metric tons.

57. The wheels of a bicycle are 28 in. in diameter. The sprocket wheel connected with the pedals has 18 sprockets; the other, 8. How many miles an hour does the rider make for one revolution of the pedals per sec.? If he travels 15 mi. per hr., how many revolutions of the pedals per min.?

58. Milk gives about 20% in weight of cream, and cream gives about 30% in weight of butter. How many liters of milk will produce 100 kg of butter, and how many kilograms of butter from 100 l of milk? The density of milk is 1.03.

59. At a certain school rain fell one day to the depth of 36 mm. Calculate the volume of water which fell upon the school yard, a hectare in area; also the weight of this volume of water; also the respective weights of the oxygen and hydrogen contained, knowing that water is formed of eight parts in weight of oxygen to one of hydrogen.

60. A cubic foot of water weighs 1000 oz., and in freezing expands $\frac{1}{40}$ of itself in length, breadth, and thickness; find the weight of a cubic foot of ice, correct to 0.1.

61. A liter of good milk weighs 1.030 kg. A milkman furnished 4.5 l of milk weighing 4.59 kg. Was there any water in it, and if so, how much?

62. How long will it take a man to walk around a square field whose area is $5\frac{5}{8}$ acres, at the rate of a mile in $10\frac{2}{3}$ mins.?

63. Having found the average number of inches of rainfall per yr. in your vicinity, determine the average number of gallons of water that fall upon the roof of your school building in one year.

64. Of the world's supply of wool in a certain year, 2,456,733,600 lbs., Great Britain produced 147 lbs. to the continent of Europe's 640 and North America's 319; North America produced 6 lbs. to Australasia's 11; Great Britain produced 15 lbs. to the Cape of Good Hope's 13; Australasia produced 22 lbs. to the River Plate's 15; Australasia produces 577,500,000 lbs; how many pounds (correct to 100,000) were produced by all other countries together?

65. The value of the food-fishing industry in Alaska and Massachusetts together in a certain year was $8,149,987, Massachusetts exceeding Alaska by $3,547,877; what was the value in each?

66. A gas holder is to be constructed in the form of a circular cylinder, such that the radius of the circle shall be equal to the height; find its dimensions to contain 100,000 cu. ft. of gas.

67. If a terrestrial globe is constructed 36 in. in diameter, find the size on its surface of the United States, whose area is 3,500,000 sq. mi., the earth's diameter being 8000 mi.

68. If 100 lbs. of copper are drawn into 1 mi. of wire, find the diameter, copper being 8.9 times as heavy as water and 1 cu. ft. of water weighing 1000 oz. avoirdupois.

69. An india rubber band 8 in. long, $\frac{3}{8}$ in. wide, $\frac{1}{16}$ in. thick, is stretched until it is 18 in. long and $\frac{1}{4}$ in. wide; what is then its thickness, assuming the volume to remain constant?

70. What is the amount of pressure exerted against one side of the upright gate of a canal, the gate being 24 ft. wide and submerged to the depth of 10 ft.?

71. A locomotive traveled for 32 secs. on a certain railroad at the rate of 112 mi. per hr.; how many revolutions were made in this time by the driving wheels, which were 78 in. in diameter?

72. If a man whose body has a surface of 15 sq. ft. dives in fresh water to the depth of 70 ft., what pressure does his body sustain?

73. Sea-water weighing 64.05 lbs. per cu. ft., what is the pressure per sq. in. at the depth of 4655 fathoms of 6 ft.?

74. What length of paper $\frac{3}{4}$ yd. wide will be required to cover a wall 15 ft. 8 in. long by 11 ft. 3 in. high, no allowance being made for matching?

75. The diagonals of a quadrilateral field are 24 chains and 35 chains in length respectively, and are perpendicular to one another; how many acres in the field?

76. How many degrees in an arc equal in length to the radius of the circle?

77. A circle is inscribed in a square, the radius of the circle being 8.5 in.; find the area between the sides of the square and the circumference of the circle.

78. Find the difference between the area of a circle 15.4 in. in diameter and that of a regular inscribed hexagon. (The side of a regular inscribed hexagon equals the radius of the circle.)

79. Three circles each 4 ft. in diameter touch one another; find the area of the triangular figure enclosed by them.

80. What is the length of the edge of the largest cube that can be cut out of a sphere 1 ft. in diameter?

81. What is the length of the edge of a cube whose surface is 9 sq. ft. 54 sq. in.?

82. A bar of metal 9 in. wide, 2 in. thick, and 8 ft. long weighs 1 lb. per cu. in.; find the length and thickness of another bar of the same metal, and of the same width and volume, if 2 in. cut off from the end weighs 27 lbs.

83. For a period of a week note carefully the number of minutes spent in the preparation of each of your various lessons. Represent the averages for the various subjects graphically.

84. Gunpowder being composed of $\frac{1}{8}$ sulphur, 75% niter, and the balance charcoal, how many pounds of each in 200 lbs. of powder?

85. Three persons contribute \$1000, \$1200, \$1780 respectively, and after trading 15 yrs. dissolve partnership; the firm then being worth \$18,000, what did each man receive?

86. A piece of wood which weighs 70 oz. in air has attached to it a piece of copper which weighs 36 oz. in air and 31.5 oz. in water; the united mass weighs 11.7 oz. in water; what is the specific gravity of the wood?

87. Find the weight of 10 mi. of steel wire 0.147 in. in diameter, the specific gravity being 7.872.

88. Find the weight of a cast iron cylinder 8 ft. long, with a radius of 3.5 in., assuming the specific gravity to be 7.108.

89. What is the weight of a circular plate of copper 11 in. in diameter and $\frac{5}{8}$ in. thick, copper weighing 549 lbs. per cu. ft.?

90. What is the weight of a slate blackboard 19.5 ft. long, 3.5 ft. wide, and $\frac{5}{8}$ in. thick, the specific gravity of the slate being 2.848?

91. If 31 cm^3 of gold weighs 599 g, find the specific gravity of gold.

92. Find the time between 3 and 4 o'clock when the hour and minute hands of a watch are (1) together, (2) opposite, (3) at right angles to one another.

93. A locomotive is going at the rate of 45 mi. per hr.; how many revolutions does the drive-wheel, 22 ft. in circumference, make in a second?

94. How many seconds will a train 184 ft. long, traveling at the rate of 21 mi. per hr., take in passing another train 223 ft. long, going in the same direction at the rate of 16 mi. per hr.? how many seconds if they are going in opposite directions?

95. How many telegraph poles 58 ft. apart will a traveler by train going at the rate of 48 mi. per hr. pass in a minute?

96. A train leaves C for M at 9 A.M., traveling at a uniform rate of 15 mi. per hr.; an express train leaves M for C at 10 A.M. at 40 mi. per hr.; at what time will they meet, and at what distance from C, the distance from C to M being 50 mi.?

97. A sledge party travels northward on an ice-floe at the rate of 12 mi. per da.; the floe is itself drifting eastward at the rate of 100 rods per hr.; at what rate is the sledge really moving?

98. A starts out on a bicycle at the rate of 8 mi. per hr.; after he has gone 3 mi., B follows at the rate of 10 mi. per hr.; after how many hours will B overtake A?

99. Out of a circle 18 in. in diameter there is cut a circle 13.5 in. in diameter; what per cent of the original circle is left?

100. A laborer asks to have his time changed from 10 hrs. to 8 hrs. a day without decrease of daily pay; by what per cent of his hourly wages does he ask them to be increased?

101. There was formerly in use a discount known as "true discount," which was the interest on the present worth of a given amount, that is, on such a sum as placed at interest for the given time should equal the given amount; show that the bank discount equals the true discount plus the interest on the true discount.

102. What is the present worth of $1356.80 due in 1 yr. 4 mo., the rate being 4½%?

103. What is the present worth and true discount of $1120 due in 2 yrs. at 6%?

104. What is the present worth and true discount of $1000 due in 11 mo. at 5%?

105. What is the present worth and true discount of $1430.40 due in 16 mo., the rate being 3½%?

106. A dealer sells a machine for $80, taking a note to be paid without grace in 8 equal monthly payments without interest; after two payments he takes the note to a bank and discounts it at 6%; find the proceeds.

107. A dealer sells a bicycle for $50, taking a note to be paid without grace in 10 equal monthly payments without interest; after 4 payments he discounts the note at 5%; find the proceeds.

108. A man sells a lot for $500, taking a note to be paid without grace in 10 equal monthly payments with interest at 5%; after half of the payments have been made he discounts the note at 6%; find the proceeds.

109. A bookseller agrees to furnish a certain number of books for $66.30, after giving a discount of 15% upon the list prices. He himself gets a discount of 25%. What is his gain?

110. A house cost $5000 and rents for $25 a month, with $25 to pay annually for repairs and $50 for taxes; what is the difference in the income from this and from the same money invested in 6% stock at 96?

111. A certain set of books in 10 volumes is offered for $66.60 cash, or $69 payable $5 cash and $5 each succeeding month until the total amount of $69 has been paid. Which is the cheaper arrangement, money being worth 6%?

112. A man invests $\frac{2}{5}$ of his money at 4% and the rest at $4\frac{1}{2}$%. His annual income is $997.60. What is the ratio between the two parts of his income? What are the two amounts invested? What is the average interest upon his capital?

113. On a mortgage for $1700 dated May 28, 1900, there was paid Nov. 12, 1900, $80; Sept. 20, 1901, $314; Jan. 2, 1902, $50; Apr. 17, 1902, $160; what was due Dec. 12, 1902, at 6%?

114. Which investment yields the better rate of income, one of $4200, yielding $168 semi-annually, or one of $7500, producing $712.50 annually?

115. If an agent's commission is $290.40 when he sells $11,606 worth of goods, how much would it be when he sells $7416 worth?

116. What is the difference on a bill of $1750 between a discount of 40% and a discount of 30% 10%?

117. What per cent on the cost is gained by selling goods at the list price, they having been purchased at "a quarter off"?

118. How long will it take a sum of money to double itself at 6% simple interest? at 6%, compounded semi-annually? at 6%, compounded quarterly? (Answer correct to 0.001.)

119. The cost of maintaining the life-saving service of the United States for a certain year was $1,345,324; the value of the property saved was $9,145,000, and the value of the property involved was $10,647,000; these values were what per cent of the cost of maintaining the service?

120. The average number of hours which it takes for a letter to go from New York to London is 162.5 by one route and 176.7 by another; what per cent is gained by taking the shorter passage?

121. The total valuation of farms in the United States in a certain year was $13,279,252,649, of the implements $494,247,467, and of the live stock $2,208,767,573; the value of farm products was $2,460,107,454. The value of the products was what per cent of the total valuation of farms, implements, and live stock?

122. After decreasing 13% from the acreage in a certain year, the number of acres in corn in the United States the following year was 62,671,724; what was the acreage in the former year, correct to 1000?

123. The United States produced 4,019,995 tons of steel in a certain year, and that produced by other countries constituted 65.26% of the total production of the world; required the total production.

124. In a certain year the combined capital of all the fire insurance companies of the United States was $70,225,220; the total income for

that year was $175,749,635, the amount paid for losses $89,212,971, for expenses and surplus $54,203,408, the balance going to dividends; what was the average rate of dividend?

125. 59.9% of the total production of copper in the world comes from outside the United States; how many tons does this represent, the production of the United States being 105,774 tons annually?

126. The dividends paid by the National Banks in a certain year were $45,333,270, the capital being $672,951,450; what was the average rate of dividend? (Answer to 0.1%.)

127. Find x, given (*a*) $5^x = 20$; (*b*) $100^x = 2$; (*c*) $8^x = 100$; (*d*) $4^{x+1} = 50$.

128. Show that $\log(1 + 2 + 3) = \log 1 + \log 2 + \log 3$. Is this true for $1 + 2$? for $1 + 2 + 3 + 4$?

129. Why is 10 the most practical base for a system of logarithms? Why can 1 not be the base of a system?

130. What is the logarithm of 125 to the base 5? of 729 to the base 3? of 64 to the base 4? to the base 2?

131. Show that if 3 is the base of the system of logarithms, $\log 81 + \log 243 = \log 19683$, by finding each logarithm.

132. We write numbers on a scale of 10; that is, we write up to 10, then to $2 \cdot 10$, then to $3 \cdot 10$, ... $10 \cdot 10$ ..... Show that we might write on a scale of 9 using one less digit; or on a scale of 8 using two less digits, and so on to a scale of 2.

133. How many characters are necessary for the scale of 10? of 8? of 12? of 100? of n?

134. What characters are needed in the scale of 2? Write 15 on the scale of 2.

135. Write the numbers 12, 144, 145, 155, 1728, 1738 on the scale of 12. (The letters *t*, *e* may be taken to stand for 10 and 11.)

136. Add 1164 and 2345 on the scale of 7.

137. From 3542 take 1164 on the scale of 7.

138. Multiply 3542 by 4 on the scale of 7.

139. Write 25 and its nth power, on the scale of 5.

140. Write the numbers 15 and 21 on the scale of 2; multiply one by the other and hence show that if we used the scale of 2 it would not be necessary to learn the multiplication table.

141. On the scale of 10 one-half is written 0.5; on the scale of 12 it is written 0.6. Write the fractions $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, ... $\frac{1}{9}$ as decimals; also write them on the scale of 12, and hence show that the scale of 12 is adapted to computation better than the scale of 10.

APPENDIX.

NOTE I, to p. 36. THE THEORY OF SQUARE ROOT CONTINUED.

Trial divisor. The expression $2f$ is often called the *trial divisor*, $2f+n$ being called the *complete divisor*. It will be noticed in the example on p. 36 that the second trial divisor, $2f_2$, equals the sum of the first complete divisor, $2f_1+n_1$, and n_1. In other words, the new trial divisor can always be found by adding n to the last complete divisor.

For $2f_1+n_1$, added to n_1, equals $2(f_1+n_1)$.
But $f_1+n_1=f_2$, $\therefore 2(f_1+n_1)=2f_2$.

In the extraction of the square root of a long number like 299,066.7969, the ordinary abridged process may be still further shortened. In this example, it will be noticed that after the first three figures of the root have been found the next two can be found by merely dividing the remainder 950.79 by the trial divisor.

```
              5  4  6. 8  7                      0.87
             29'90'66.79'69             1092 ) 950.79
104           4 90                               77.19
1086            74 66                              .75
1092.8           9 50.79
1093.67             76.55 69
                           0
```

That this is true in general in the case of a perfect square will now be proved.

Abridgment theorem. After $n+1$ figures of the square root of a perfect second power (or square) have been found, the next n figures can be found by dividing the next remainder by the next trial divisor.

1. Let $\sqrt{s}=f+x$, where s is a perfect second power, f contains $n+1$ figures already found, and x contains n figures to be found.
2. $\because$ the $n+1$ figures of f are followed by the n figures of x, f has the value of a number of $2n+1$ figures.
3. From 1, $s=f^2+2fx+x^2$, or $\frac{s-f^2}{2f}=x+\frac{x^2}{2f}$. That is, if the remainder $s-f^2$ is divided by the trial divisor $2f$, the quotient is x plus the fraction $\frac{x^2}{2f}$.
4. $\therefore$ if it is shown that $\frac{x^2}{2f}$ is a *proper* fraction, the *integral* part of $x+\frac{x^2}{2f}$ is x, the remaining part of the root, and the theorem is proved.
5. $\because$ x contains n figures, $\therefore x < 10^n$, and $x^2 < 10^{2n}$.
6. $\because$ f has the value of a number of $2n+1$ figures, $\therefore f \nless 10^{2n}$, and $2f \nless 2 \cdot 10^{2n}$.
7. $\therefore \frac{x^2}{2f} < \frac{10^{2n}}{2 \cdot 10^{2n}}$ or $\frac{1}{2}$, a proper fraction.

Exercise. Show that the abridgment theorem holds only for a perfect second power as stated, by considering the cases of 152,399,000, and 152,399,025, the latter being a perfect second power.

NOTE II, to p. 40. THE THEORY OF CUBE ROOT CONTINUED.

Trial divisor. The expression $3f^2$ is often called the *trial divisor*, $3fn+n^2$ being called the *correction*, and $3f^2+3fn+n^2$ the *complete divisor*. It will be noticed in the example on p. 39 that the second trial divisor, $3f^2$ (or 780,300), equals the sum of the first complete divisor ($3f^2+3fn+n^2$, or 765,100) and the correction ($3fn+n^2$, or 15,100) and the square of n (n^2, or 100). This is always true in cube root.

For $f_2 = f_1 + n_1$.

$\therefore 3f_2^2 = 3(f_1 + n_1)^2 = 3f_1^2 + 6f_1n_1 + 3n_1^2$
$= 3f_1^2 + 3f_1n_1 + n_1^2$ (the preceding complete divisor)
$+ 3f_1n_1 + n_1^2$ (the correction)
$+ n_1^2$ (the square of n).

And the same reasoning holds for all trial divisors in cube root.

This is much shorter than the operation of squaring f and multiplying by 3 each time.

Abridgment theorem. After $n + 2$ figures of the cube root of a perfect third power have been found, the remaining n figures can be found by dividing the next remainder by the next trial divisor.

1. Let $\sqrt[3]{t} = f + x$, where t is a perfect third power, f contains $n + 2$ figures already found, and x contains n figures to be found.

2. $\because$ the $n + 2$ figures of f are followed by the n figures of x, f has the value of a number of $2n + 2$ figures.

3. From 1, $t = f^3 + 3f^2x + 3fx^2 + x^3$,

$\therefore \dfrac{t - f^3}{3f^2} = x + \dfrac{x^2}{f} + \dfrac{x^3}{3f^2}$. That is, if the remainder $t - f^3$ is divided by the trial divisor $3f^2$, the quotient is x plus the fractions $\dfrac{x^2}{f}$ and $\dfrac{x^3}{3f^2}$.

4. $\therefore$ if it is shown that $\dfrac{x^2}{f} + \dfrac{x^3}{3f^2}$ equals a *proper* fraction, the *integral* part of the quotient is x and the theorem is proved.

5. $\because x$ contains n figures, $\therefore x < 10^n$, $x^2 < 10^{2n}$, and $x^3 < 10^{3n}$.

6. $\because f$ has the value of a number of $2n + 2$ figures,

$\therefore f \nless 10^{2n+1}$, $f^2 \nless 10^{4n+2}$, and $3f^2 \nless 3 \cdot 10^{4n+2}$.

7. $\therefore \dfrac{x^2}{f} < \dfrac{10^{2n}}{10^{2n+1}}$, or $\dfrac{1}{10}$, and

$\dfrac{x^3}{3f^2} < \dfrac{10^{3n}}{3 \cdot 10^{4n+2}}$, or $\dfrac{1}{3 \cdot 10^{n+2}}$.

8. $\therefore \dfrac{x^2}{f} + \dfrac{x^3}{3f^2} < \dfrac{1}{10} + \dfrac{1}{3 \cdot 10^{n+2}}$, a proper fraction.

Exercise. Show that with the first three figures of the square root of 14,696,712,600,000,000, obtained by the ordinary process, the next two cannot be correctly found by division. (Similar conditions evidently exist in the theory of cube root.)

Three cube roots. (Omit if the class has not studied quadratic equations and imaginaries.) Just as 4 has two square roots, $+2$ and -2, so 8 has three cube roots, 2, $-1+\sqrt{-3}$, and $-1-\sqrt{-3}$, as may easily be proved by cubing. So in general, every number has three and only three cube roots.

1. For let $x^3 = n$; then if the value of x is found the cube root of n is known.
2. From 1, $x^3 - n = 0$.
3. $\therefore (x - \sqrt[3]{n})(x^2 + x\sqrt[3]{n} + \sqrt[3]{n^2}) = 0$, by factoring (2).
4. This equation is satisfied if either
$$x - \sqrt[3]{n} = 0, \quad \text{or} \quad x^2 + x\sqrt[3]{n} + \sqrt[3]{n^2} = 0.$$
5. If $x - \sqrt[3]{n} = 0$, then $x = \sqrt[3]{n}$, the ordinary arithmetical cube root.
6. If $x^2 + x\sqrt[3]{n} + \sqrt[3]{n^2} = 0$, then, by solving the quadratic equation,
$$x = -\tfrac{1}{2}\sqrt[3]{n} + \tfrac{1}{2}\sqrt{\sqrt[3]{n^2} - 4\sqrt[3]{n^2}}$$
$$= \sqrt[3]{n}\left(-\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{-3}\right).$$
7. That is, the three cube roots of n are
$$\sqrt[3]{n}, \quad \sqrt[3]{n}\left(-\tfrac{1}{2} + \tfrac{1}{2}\sqrt{-3}\right), \quad \sqrt[3]{n}\left(-\tfrac{1}{2} - \tfrac{1}{2}\sqrt{-3}\right).$$

Exercises on the three cube roots. 1. What are the three cube roots of 1? Verify your answer by cubing.

2. Show that each of the imaginary cube roots of 1 is the square of the other.

3. Show that the sum of the three cube roots of any number is zero.

4. Show that the product of the three cube roots of any number is that number.

5. Show that any number has four fourth roots.

NOTE III, to p. 146. THE THEORY OF COMPOUND INTEREST CONTINUED.

The theory of compound interest affords an application of logarithms for those who have studied Chap. XI. The work on p. 193, excepting steps 1–4 and Ex. 1, should be omitted by others.

E.g., by logarithms the computation on p. 145 becomes simple:

$$
\begin{aligned}
\log 1.02 &= \underline{0.0086}\\
6\cdot\log 1.02 &= 0.0516\\
\log 150\ &= \underline{2.1761}\\
&\ 2.2277 = \log 168.9,
\end{aligned}
$$

as near as the result can be obtained by the small table on p. 114.

The principal formulae are deduced as follows:

1. Let $p =$ the number of dollars of principal, r the rate at which the interest is to be compounded annually, t the number of years, and $a_1, a_2, a_3, \cdots\cdots a_t$ the amounts for 1, 2, 3, ⋯⋯ t years.

2. Then,

$$
\begin{aligned}
a_1 &= p + rp = (1+r)p,\\
a_2 &= (1+r)p + r(1+r)p = (1+r)^2p,\\
a_3 &= (1+r)^3p, \text{ and in general}
\end{aligned}
$$

3. $$a_t = (1+r)^t p.$$

4. $$\therefore p = \frac{a_t}{(1+r)^t}.$$

5. From 3, $\log a_t = t \log(1+r) + \log p$.

6. $$\therefore t = \frac{\log a_t - \log p}{\log(1+r)}.$$

7. And from 5, $r = \text{antilog}\,\dfrac{\log a_t - \log p}{t} - 1$.

If the interest is compounded semi-annually for t yrs. at $r\%$ a year, the amount is evidently the same as if the interest were compounded annually for $2t$ yrs. at $\frac{r}{2}\%$ a year.

Exercises. 1. What is the amount of each of the following?

	FACE.	TIME.	RATE.	COMPOUNDED.
(*a*)	$250	3 yrs. 2 mo. 10 da.	4%	semi-annually.
(*b*)	$75	1 yr. 4 " 3 "	4%	quarterly.

2. Find the principal, given the

	AMOUNT.	TIME.	RATE.	COMPOUNDED.
(*a*)	$384.03	5 yrs.	5%	semi-annually.
(*b*)	$162.36	2 "	4%	"

3. In how many years will $200 amount to $310.26 at 5%, compounded annually?

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